Rosenberg's reconstruction theorem

A path to noncommutative algebraic geometry

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In this essay we introduce Rosenberg's reconstruction theorem, which asserts that two suitably nice schemes X and Y are isomorphic if and only if their associated categories of quasicoherent sheaves $\mathrm{QCoh}(X)$ and $\mathrm{QCoh}(Y)$ are equivalent. We also explain a strengthening of this result which fully classifies the structure of an arbitrary equivalence $F : \mathrm{QCoh}(X) \simeq \mathrm{QCoh}(Y)$, namely that any such functor factors as the pullback by an isomorphism $f : Y \to X$ followed by tensoring with a line bundle over Y.

We describe the structure of a proof of this result due to Brandenburg and Gabber [2] which was born out of an attempt to understand Rosenberg's original proof [19], before introducing a corresponding functor-of-points formulation. We then explain how results from the literature—principally of Brandenburg–Chirvasitu [3] and Nyman [16]—combine to give a proof of the functorial formulation, and we spell out the main ideas of the proofs of their original results. We do not aim to give a self contained exposition, instead attempting to outline the structures of proofs by analogy with the straightforward case where X and Y are both affine schemes. Indeed, in the affine case the Eilenberg–Watts theorem from commutative algebra can be applied directly to perform the heavy lifting, and we will see that one way to interpret Rosenberg's reconstruction theorem is as a consequence of a substantial generalisation of the Eilenberg–Watts theorem.

We conclude by explaining how Rosenberg's reconstruction theorem—in all of its variegated incarnations—provides an entry-point into the subject of noncommutative algebraic geometry. By the end we will also see that one partial strengthening of Rosenberg's theorem by Brandenburg–Chirvasitu [3] implies that 1-algebraic geometry can be interpreted as being "2-affine" in a suitable higher categorical sense.

1 Rosenberg's original proof

The reconstruction theorem was first proved by Gabriel [9] in the case of Noetherian schemes. In [19] Rosenberg lifted this restriction[1] and required that the schemes in question merely be quasiseparated. A modern and essentially self-contained proof by Brandenburg using the ideas of Gabber appears in [2], and we now take a moment to quickly describe the basic constructions. We make no attempt to be complete, but themes such as reduction to the affine case will recur in later sections.

Fix rings R and S. We write ${}_R\mathrm{Mod}$ for the category of left R-modules, and ${}_R\mathrm{Mod}_S$ for the category of (R,S)-bimodules. When we want to emphasise that M is an (R,S)-bimodule we sometimes write ${}_RM_S$, and when we want to emphasise the restriction

[1] The proof is long and complex, and appears as the final Proposition 10.7.1 of [19].

to only the left R-module structure we write ${}_R M$ (and analogously on the right). If M is a left R-module and $(r, m) \in R \times M$, then for emphasis we write $r \triangleright m$ for the action of r on m. If $\mathscr{C}$ is any category and $X, Y \in \mathscr{C}$ are objects then we write $\mathrm{Hom}_{\mathscr{C}}(X \to Y)$ for the collection of morphisms from X to Y in $\mathscr{C}$. We use $[\mathscr{C} \to \mathscr{D}]$ to denote the category of functors from $\mathscr{C}$ to $\mathscr{D}$, and later if we put conditions or structure on these functors we notate this over the arrow. All of our abelian categories will be cocomplete, and have exact directed colimits. Functors between abelian categories will be assumed additive. We also use Spec to denote the hom-functor from the category **CAlg** of rings to sets defined by $\operatorname{Spec} R(S) := \mathrm{Hom}_{\mathbf{CAlg}}(R \to S)$. Finally, we use Sch to denote the category of schemes and their morphisms, and $\mathrm{QCoh}(X)$ for the category of quasicoherent sheaves over a scheme X. In this section a scheme is a locally ringed space.

Definition 1.1. An object A of an abelian category $\mathscr{A}$ is[2] *spectral* if A is not zero and if $A' \hookrightarrow A$ is any nonzero subobject then A is a subquotient of a (perhaps infinite) direct sum of copies of A'.

The *spectrum* $\operatorname{Spec}(\mathscr{A})$ of $\mathscr{A}$ is the quotient of the set of spectral objects of $\mathscr{A}$ by the equivalence relation that $A \simeq A'$ exactly when A and A' are each respectively subquotients of a (possibly infinite) direct sum of copies of the other.

Remark 1.2. In later sections we will give and elect to work with another definition of the spectrum of a (suitably endowed) category, which is more categorical in nature. Nonetheless, when both notions of spectrum make sense it will follow from our results that they both agree.

The proof of the reconstruction theorem in [2] (which we follow here closely) now proceeds by showing that the spectrum of $\mathrm{QCoh}(X)$ for X a scheme is naturally a topological space, and in fact a scheme, and then that there is a natural isomorphism $X \to \operatorname{Spec}(\mathrm{QCoh}(X))$ of sets, then of topological spaces, and finally of locally ringed spaces. Of course, Rosenberg's reconstruction theorem then follows immediately.

As a basic sanity-check we note the following lemma.

Lemma 1.3. *Let R be any ring. Then the spectrum* $\operatorname{Spec}(\mathrm{QCoh}(\operatorname{Spec} R)) \simeq \operatorname{Spec}({}_R\mathrm{Mod})$ *is in canonical bijection with the prime ideals of R.*

Proof. This follows from Proposition 2.6 of [2], and is just commutative algebra. Note that by the definition of the spectrum any class $[M] \in \operatorname{Spec}({}_R\mathrm{Mod})$ is represented by the class of the submodule of M generated by some nonzero $m \in M$, so even $[M] = [R/\operatorname{Ann}_R m]$ with $R/\operatorname{Ann}_R m \neq 0$, and the only difficulty is showing that the ideal $\operatorname{Ann}_R m$ of R is prime. $\square$

Results of this kind are typical in [2], where a particular step is often established for affine schemes before being used to bootstrap the extension to the general case. For example, in the affine case a natural map from the prime ideals $\mathfrak{p}$ of a ring R to the set $\operatorname{Spec}(\mathrm{QCoh}(R)) \simeq {}_R\mathrm{Mod}$ arises by mapping a prime ideal $\mathfrak{p}$ of R to the class $[R/\mathfrak{p}]$ (inspired by the previous lemma). Given a general scheme X and a point $x \in X$, if $\mathscr{I}_x \subset \mathscr{O}_X$ is a quasicoherent ideal canonically corresponding to x then we have the analogous mapping

$$x \mapsto [\mathscr{O}/\mathscr{I}_x]. \tag{1}$$

[2] Brandenburg explains in [2] that this definition contains an important modification of Rosenberg's original definition due to Ofer Gabber.

Now, for any inclusion of a full subcategory of an abelian category $\mathscr{B} \subseteq \mathscr{A}$, we can canonically build the subset $\mathrm{Spec}^{\mathscr{A}}(\mathscr{B}) := \{[M] \in \mathrm{Spec}(\mathscr{A}) : M \in \mathscr{B}\}$ of $\mathrm{Spec}(\mathscr{A})$. In fact, the set $\mathrm{Spec}(\mathscr{A})$ is naturally made into a topological space by declaring that the closed sets are $\mathrm{Spec}^{\mathscr{A}}(\mathscr{B})$ for $\mathscr{B} \subseteq \mathscr{A}$ any so-called *reflexive topologizing subcategory* [2]. The required conditions on $\mathscr{B}$ are merely first that $\mathscr{B}$ is closed under taking subquotients and direct sums (*topologizing*), and second that the inclusion of $\mathscr{B}$ into $\mathscr{A}$ is a right-adjoint (*reflective*).

In analogy with Lemma 1.3, again in the affine case we have the following concrete module-theoretic description of the induced topology.

Lemma 1.4 (Proposition 3.4 of [2]). *The topologizing reflexive subcategories of* ${}_R\mathrm{Mod}$ *are in natural correspondence with the ideals of R, where an ideal $I \subset R$ corresponds to the full subcategory of* ${}_R\mathrm{Mod}$ *consisting of the modules annihilated by I.*

The proof now proceeds by showing that (1) is a continuous map with respect to this topology. After this, we show that when X is quasiseparated the global sections $\Gamma(\mathscr{O}_X)$ of its structure sheaf are recovered as the ring of endomorphisms of the identity functor $\mathrm{QCoh}(X) \to \mathrm{QCoh}(X)$. Local sections of $U \hookrightarrow X$ are recovered as the endomorphisms of the identity functor on a quotient of $\mathrm{QCoh}(X)$ by the so-called *thick* subcategory of $\mathrm{QCoh}(X)$ of quasicoherent sheaves which vanish over U [2, 9]. Going via the endomorphism construction one finally obtains an isomorphism $X \to \mathrm{Spec}(\mathrm{QCoh}(X))$ of locally ringed spaces, as desired.

Thus we arrive at the following generalized theorem of Rosenberg [19].

Theorem 1.5 (Rosenberg's reconstruction theorem). *Every quasiseparated scheme X is isomorphic as a locally ringed space to the spectrum* $\mathrm{Spec}(\mathrm{QCoh}(X))$. *Equivalences* $\mathrm{QCoh}(X) \simeq \mathrm{QCoh}(Y)$ *induce isomorphisms* $\mathrm{Spec}(\mathrm{QCoh}(X)) \cong \mathrm{Spec}(\mathrm{QCoh}(Y))$ *and therefore also $X \cong Y$.*

In [2] Brandenburg actually classifies all equivalences $\mathrm{QCoh}(X) \simeq \mathrm{QCoh}(Y)$ and obtains the next theorem (note that line bundles, i.e. locally free sheaves of rank 1, are equivalently invertible sheaves). In Section 4 we will prove this extended version of Theorem 1.5 in the affine case, and then (under suitably general hypotheses) we will recover a version for schemes in Section 6.

Theorem 1.6 (Rosenberg's reconstruction theorem, extended version[3]). *The collection of equivalences* $\mathrm{QCoh}(X) \simeq \mathrm{QCoh}(Y)$ *for X and Y quasiseparated schemes is itself equivalent as a category to the (discrete) category of pairs*

$$\Big\{(f : Y \to X, \mathscr{L} \in \mathrm{QCoh}(Y)) \mid f \text{ an isomorphism and } \mathscr{L} \text{ a line bundle}\Big\}.$$

A pair $(f, \mathscr{L})$ maps to the equivalence $- \mapsto f^*(-) \otimes \mathscr{L}$ under this correspondence.

2 The functor of points perspective

Henceforth our schemes will represented by their functors of points. In contrast to the previous section it is now our objective to formulate a version of the spectrum which fits in naturally with functors of points. The first step will be to define a "categorical" spectrum, which we do below.

[3] This is Theorem 5.4 of [2].

Now, if $\mathscr{C}$ is any category let $K_0(\mathscr{C})$ denote the set of isomorphism classes of the objects of $\mathscr{C}$. Let Cat denote the 2-category of categories and likewise let $\mathrm{AbCat}_{\otimes}$ denote the 2-category of monoidal abelian categories (with monoidal and abelian structures which we always assume are compatible). Our monoidal categories will always be symmetric monoidal, and our monoidal functors will be strong (i.e. not merely lax) unless explicitly specified. Moreover if $F : \mathscr{C} \to \mathscr{D}$ is a monoidal functor between monoidal categories we denote its identity isomorphism (*identitor*) by $\iota : \mathbb{1}_{\mathscr{D}} \to F(\mathbb{1}_{\mathscr{C}})$ and its multiplication isomorphism (*tensorator*) by $J_{C,D} : F(C) \otimes F(D) \to F(C \otimes D)$.

Definition 2.1. Let $\mathscr{A}$ be a cocomplete[4] abelian monoidal category. The *categorical spectrum* of $\mathscr{A}$ is the 2-functor Sch $\to$ Cat defined for each $Y \in \mathrm{Sch}$ by the assignment

$$\mathbf{Spec}(\mathscr{A})(Y) := \left\{ \text{cocontinuous monoidal functors } \mathscr{A} \to \mathrm{QCoh}(Y) \text{ and their morphisms} \right\}.$$

In particular the morphisms of $\mathbf{Spec}(\mathscr{A})(Y)$ are the monoidal natural transformations. For our purposes we will often want to restrict to the truncated *spectrum* 1-functor

$$\underline{\mathrm{Spec}}(\mathscr{A})(Y) := K_0(\mathbf{Spec}(\mathscr{A})(\mathrm{QCoh}(Y))),$$

which simply reports a set for each $Y \in \mathrm{Sch}$.

Finally, we often implicitly think of a ring $R \in \mathbf{CAlg}$ as its spectrum $\mathrm{Spec}\, R$, and thus by convention $\underline{\mathrm{Spec}}(\mathscr{A})(R)$ makes sense for any cocomplete abelian monoidal category $\mathscr{A}$ and ring R.

In this language, a proof of Rosenberg's reconstruction theorem naturally fits into five steps:

Step 1. Show that for suitably nice (quasicompact quasiseparated) schemes X, there is a natural isomorphism in $R \in \mathbf{CAlg}$:

$$X(R) \cong \underline{\mathrm{Spec}}(\mathrm{QCoh}(X))(R).$$

This will follow as a consequence of Theorem 5.1, which is Brandenburg–Chirvasitu's main result of [3].

Step 2. It follows immediately that if $\mathrm{QCoh}(X)$ and $\mathrm{QCoh}(Y)$ are monoidally equivalent (and both X and Y are "nice enough") then

$$\underline{\mathrm{Spec}}(\mathrm{QCoh}(X)) \cong \underline{\mathrm{Spec}}(\mathrm{QCoh}(Y))$$

and therefore $X \cong Y$. In fact, the full generality of Theorem 5.1 allows us to lift the niceness hypothesis on one of X or Y.

Step 3. Given an arbitrary (not necessarily monoidal) equivalence $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$, show that $\mathscr{L} := F(\mathscr{O}_X)$ is an invertible object of $\mathrm{QCoh}(Y)$. Consequently post-composition of F with the autoequivalence of $G : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(Y)$ which tensors with an inverse of $\mathscr{L}$ gives a functor $G \circ F$ which preserves the tensor unit.

[4] This hypothesis is not necessary for the definition of the categorical spectrum to make sense, but it is necessary for **Spec** to enjoy the property of being a stack (i.e. for Theorem 5.14 to hold below).

Step 4. Show that any equivalence $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ equipped with the data of an isomorphism $\iota : F(\mathcal{O}_X) \to \mathcal{O}_Y$ is canonically a monoidal functor.

Step 5. Conclude that by Step 2 the composite $G \circ F$ is the pullback by an isomorphism $f : Y \to X$, and therefore there is a natural isomorphism

$$F(-) \cong f^*(-) \otimes \mathcal{L},$$

thus fully classifying the equivalences $\mathrm{QCoh}(X) \simeq \mathrm{QCoh}(Y)$ when X is nice.

We embark on our journey with the humble first objective in Section 4 of executing this entire strategy in the case when $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are both affine (the key result will be Theorem 4.2). We then in Section 5 give an exposition of Brandenburg–Chirvasitu's original proof [3] that cocontinuous monoidal functors $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ between quasiseparated schemes are given by the pullback by a morphism $f : Y \to X$. In doing this we focus on bootstrapping from the affine case into the case where only Y is assumed affine, and then again to the general case. This will conclude Steps 1 and 2, whence Steps 3, 4, and 5 will follow in Section 6 from general categorical machinery which quite closely mirrors the module case.

3 Module category ingredients

We will need some module-theoretic preliminaries, which we now dispatch with.

Definition 3.1. For $R, S \in \mathbf{CAlg}$ and $B \in {}_S\mathrm{Mod}_R$ an (S,R)-bimodule, let T_B be the (cocontinuous) functor ${}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$ defined by

$$T_B : {}_R M \mapsto {}_S B \otimes_R M,$$

i.e. tensoring with B on the left.

Key to handling the affine case is the following famous theorem in commutative algebra.

Theorem 3.2 (Eilenberg–Watts theorem, [11, 8, 23])**.** *Fix $R, S \in \mathbf{CAlg}$, and let F : ${}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$ be a functor. If F is right exact and preserves small coproducts then there is a natural isomorphism $F \cong T_B$ for some $B \in {}_S\mathrm{Mod}_R$.*

Proof idea. The (S,R)-bimodule underlying F is $B := F(R)$—such B is automatically a left S-module, and so we must provide the right R-module structure. This is equivalently the data of a morphism $R \to \mathrm{End}_{{}_S\mathrm{Mod}}(F(R) \to F(R))$, which is provided by pre-composing the action of the functor F with the (famous) canonical isomorphism $R \cong \mathrm{End}_{{}_R\mathrm{Mod}}(R \to R)$. □

Remark 3.3. The category ${}_R\mathrm{Mod}$ is cocomplete and locally small. Because in addition right exact functors $F : {}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$ which preserve small coproducts satisfy a certain technical condition, the General Adjoint Functor Theorem yields that Theorem 3.2 applies to F exactly when F is cocontinuous [11]. We will often require hypotheses of cocontinuity on functors in the sequel.

Proposition 3.4. *Ring morphisms $f : R \to S$ are in canonical correspondence with (S,R)-bimodule structures on S which extend its natural left S-module structure.*

Proof. Starting with a ring morphism $f : R \to S$ then the left S-module given by S itself acquires a right R-module structure simply by the action

$$s \triangleleft r := s \cdot f(r)$$

using the multiplication in S. By associativity of S's multiplication this is actually an (S,R)-bimodule action, as desired.

Conversely, let ${}_SS_R$ be an (S,R)-bimodule structure extending the natural left S-module structure on S. The data of the right R-module structure is carried by a map $- \triangleleft - : S \times R \to S$, which by currying is the same as a ring map $R \to \mathrm{End}_{\mathrm{Set}}(S)$. Actually the bimodule associativity condition allows us to restrict the codomain to obtain a map $R \to \mathrm{End}_{{}_S\mathrm{Mod}}(S)$ (i.e. into the left S-module endomorphisms of S). But $\mathrm{End}_{{}_S\mathrm{Mod}}(S) \cong S$ canonically, so we obtain the desired ring map $f : R \to S$.

It is clear that these constructions are mutually inverse, so this completes the proof. □

Proposition 3.5. *Let $B \in {}_S\mathrm{Mod}_R$ be an (S,R)-bimodule. Isomorphisms $\iota : T_B(R) \to S$ canonically determine (strong) monoidal structures on T_B, and vice versa.*

Proof. Obviously any monoidal structure on $T_B : {}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$ determines such an isomorphism ι by simply forgetting the tensorator.

On the other hand if $\iota : T_B(R) \to S$ is any isomorphism then

$${}_SB \cong {}_SB \otimes_R R = T_B(R) \cong {}_SS$$

as (left) S-modules. By transport of structure S then inherits the structure of an (S,R)-bimodule which extends its canonical left S-module structure, and there is a natural isomorphism of functors $T_B \cong T_{{}_SS_R}$. Since every left R-module is presentable as a direct sum of copies of the tensor unit R, observe that we can compute $F(M \otimes N)$ for $F : {}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$ any cocontinuous monoidal functor by distributing F over a presentation of M and N and then using the composite of isomorphisms $F(R \otimes R) \to F(R) \xrightarrow{\iota} S$ to obtain a presentation in ${}_S\mathrm{Mod}$.[5] This implies that the natural isomorphism $T_B \cong T_{{}_SS_R}$ is actually monoidal.

We can now assume that B is just such a bimodule ${}_SS_R$ (and we will drop the subscripts on S for now on). We claim that T_S has a canonical tensorator $J : T_S({}_RM) \otimes_S T_S({}_RN) \to T_S(M \otimes_R N)$ compatible with the ambient identitor provided. Indeed, J is given by

$$\begin{gathered}({}_SS \otimes_R M) \otimes_S ({}_SS \otimes_R N) \to {}_SS \otimes_R (M \otimes_R N), \\ (s \otimes m) \otimes (s' \otimes n) \mapsto ss' \otimes (m \otimes n),\end{gathered}$$

which is certainly an isomorphism. The associativity constraint and identity constraints for J follow from the same property of the tensor product of modules themselves. □

Corollary 3.5.1. *Using the notation of the previous proposition, strong monoidal functor structures (T_B, ι, J) canonically correspond to pairs (f, g) with $f : R \to S$ a ring morphism and $g : {}_SB \to {}_SS$ an isomorphism of left S-modules.*

[5] See comments by Eric Wofsey at `https://math.stackexchange.com/questions/3936887/special-case-of-the-eilenberg-watts-theorem-for-the-base-ring`.

Proof. As in Proposition 3.5 we obtain the isomorphism $g : {}_S B \to {}_S S$ from ι, and then by Proposition 3.4 the induced bimodule structure on S corresponds uniquely to a ring morphism $f : R \to S$, as desired. On the other hand a ring morphism $f : R \to S$ equips S with a right R-module structure extending its left S-module structure, and then an isomorphism ${}_S B \to {}_S S$ permits transport of this right R-module structure to B. We then obtain a strong monoidal functor structure on T_B as above. □

Proposition 3.6. *Let $B, B' \in {}_S\mathrm{Mod}_R$ be bimodules and let (T_B, ι, J) and $(T_{B'}, \iota', J')$ be strong monoidal functors. Then T_B and $T_{B'}$ are monoidally naturally isomorphic*[6] *if and only if they have equal corresponding ring morphisms $R \to S$ obtained from Corollary* 3.5.1.

Proof. We can build a bimodule isomorphism $B \cong B'$ from the knowledge that T_B and $T_{B'}$ correspond to the same morphisms by simply tracing through the constructions which we performed above, since then in particular the hypotheses imply that B and B' are isomorphic to the same bimodule ${}_S S_R$. □

By the Theorem 3.2 (the Eilenberg–Watts theorem) we can equivalently state Proposition 3.6 as an equivalence between the (discrete category) $\mathrm{Hom}_{\mathbf{CAlg}}(R \to S) \cong \mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}\, S \to \mathrm{Spec}\, R)$ and the category of monoidal functors $[\mathrm{QCoh}(\mathrm{Spec}\, R) \xrightarrow{\otimes} \mathrm{QCoh}(\mathrm{Spec}\, S)]$. This foreshadows the situation to come.

4 Rosenberg's theorem: affine case

We are now ready to prove Rosenberg's reconstruction theorem for affine schemes.

Lemma 4.1. *Let R be a ring. Every invertible left R-module M canonically induces an autoequivalence of the category ${}_R\mathrm{Mod}$.*

Proof. For each R-module M whatever there is a cocontinuous functor $V_M : {}_R\mathrm{Mod} \to {}_R\mathrm{Mod}$ arising from the tensor product simply via

$$ {}_R N \mapsto M \otimes_R N. $$

If N is any other R-module then observe that the composite $V_N \circ V_M$ is naturally isomorphic to $V_{N \otimes_R M}$. Moreover each morphism $\phi : M \to N$ of R-modules induces a natural transformation $\eta_\phi : V_M \to V_N$.

In particular, if N is an inverse of M then the witnessing isomorphism $\phi : N \otimes_R M \to R$ gives rise to a chain of natural isomorphisms

$$ V_N \circ V_M \cong V_{N \otimes_R M} \cong V_R \cong \mathrm{id}_{{}_R\mathrm{Mod}}, $$

with the last isomorphism coming from the left unitor of ${}_R\mathrm{Mod}$. The symmetric braiding of ${}_R\mathrm{Mod}$ gives a natural isomorphism $V_N \circ V_M \cong V_M \circ V_N$, and so we also have $V_M \circ V_N \cong \mathrm{id}_{{}_R\mathrm{Mod}}$. Therefore V_M is part of an autoequivalence (as witnessed by V_N), as desired. □

[6] Actually, the hypothesis of monoidal natural isomorphism can be weakened to requiring the mere existence of a monoidal natural transformation $T_B \to T_{B'}$ (see Proposition 3.1.1 of [2]), but we will not require this for our purposes.

Theorem 4.2 (Rosenberg's reconstruction theorem, affine version)**.** *Fix rings R and S. There is a canonical bijection between the sets*

$$\Big\{ (f : R \to S, [L] \in K_0({}_S\mathrm{Mod})) \mid f \text{ is an isomorphism and } L \text{ is invertible} \Big\}$$

and

$$\underline{\mathrm{Spec}}({}_R\mathrm{Mod})(S) \cong \Big\{ F : {}_R\mathrm{Mod} \to {}_S\mathrm{Mod} \mid F \text{ is an equivalence} \Big\} / \sim$$

where the relation $\sim$ is the quotient by natural isomorphism.

Moreover, pairs $(f : R \to S, [L] \in K_0({}_S\mathrm{Mod}))$ with f an isomorphism and L isomorphic as an S-module to S itself correspond under this bijection to those equivalences which possess monoidal structures.

Proof. Observe that equivalences of categories are certainly cocontinuous, so by the Eilenberg–Watts theorem an equivalence $F : {}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$ is, up to natural isomorphism, given by T_B for an (S,R)-bimodule B. But a pseudoinverse $G : {}_S\mathrm{Mod} \to {}_R\mathrm{Mod}$ to F is also automatically cocontinuous, so therefore is also given up to isomorphism by T_A for an (R,S)-bimodule A. Since the composites FG and GF must be naturally isomorphic to the identity, the explicit descriptions of these functors provided by the Eilenberg–Watts theorem yields isomorphisms of modules

$${}_RA \otimes_S ({}_SB \otimes_R R) \cong {}_RA \otimes_S B \cong {}_RR \quad \text{and} \quad {}_SB \otimes_R A \cong {}_SB \otimes_R ({}_RA \otimes_S S) \cong {}_SS. \tag{2}$$

Concentrating on the former isomorphism, we find that the S-module A_S witnesses the fact that ${}_SB_R$ is invertible as a left S-module.

Next, we want to precompose with the autoequivalence T_C of ${}_S\mathrm{Mod}$ which tensors with $C := {}_SS \otimes_R A_S$ (as verified in Lemma 4.1), thereby establishing that the composite $T_C \circ T_B$ sends R to an object of ${}_S\mathrm{Mod}$ isomorphic to S. Therefore our previous result Corollary 3.5.1 engages to yield that the entire composite $T_C \circ T_B$ corresponds to the data of a single ring morphism $f : R \to S$. Repeating this process with the second isomorphism of (2), we obtain a morphism $g : S \to R$, which by functoriality[7] of our constructions is mutually inverse to f on both sides (just because F and G are themselves mutually inverse up to natural isomorphism).

Thus we obtain the data of a pair $(f, {}_SB)$ with f an isomorphism and B invertible as a left S-module. It remains to verify that f and the isomorphism class of B is respected by a natural isomorphism of functors $F \cong F'$. But such a natural transformation provides an isomorphism $F(R) \cong F'(R)$ showing that B is respected, and f is respected by monoidal natural isomorphism by Proposition 3.6 (and this is sufficient by Footnote 7).

Conversely, given a pair $(f \in R \to S, [L] \in K_0({}_S\mathrm{Mod}))$, let T_L be the autoequivalence of ${}_S\mathrm{Mod}$ induced by tensoring with L. Equip S with the canonical (S,R)-bimodule structure provided by f. Then simply set $F := T_L \circ T_B$ (clearly the isomorphism class of F is unchanged by varying L in its isomorphism class).

Finally, let $F : {}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$ be the equivalence associated to a pair $(f, [L])$ (with the former technically determined only up to natural isomorphism). Further suppose that $[L] = [S]$ with S given the canonical left S-module structure. In this situation an isomorphism $\phi : L \to S$ gives rise to an isomorphism $F(R) \cong S$, which by Proposition 3.5 determines a monoidal structure on F. Conversely if an isomorphism $F(R) \to S$ exists then $F(R) \cong L$ so $[L] = [S]$. This completes the proof. □

[7] Technically to appeal to functoriality we need to upgrade these natural isomorphisms (suitably whiskered) to monoidal natural isomorphisms. But, just as how in this case tensorators of monoidal functors are induced by identitors, it is enough to check that our natural transformations respect the identitors. This is true by construction.

Remark 4.3. It is easy to see from the proof of Theorem 4.2 that the role of the quotients by isomorphism is simply to clarify its statement. Indeed, consider the collection of quadruplets

$$(f : R \cong S, L \in {}_S\mathrm{Mod}, L' \in {}_S\mathrm{Mod}, \phi : L \otimes_S L' \cong {}_S S) \tag{3}$$

in particular with ϕ witnessing a specific isomorphism between S and L tensored with a specific choice of inverse module L'.

We can also consider quadruplets of the form

$$(F : {}_R\mathrm{Mod} \to {}_S\mathrm{Mod}, G : {}_S\mathrm{Mod} \to {}_R\mathrm{Mod}, \alpha : GF \cong \mathrm{id}_{{}_R\mathrm{Mod}}, \beta : FG \cong \mathrm{id}_{{}_S\mathrm{Mod}}), \tag{4}$$

where α and β are natural isomorphisms of functors (i.e. the full data of an equivalence of categories). We can impose an equivalence relation on this latter data by declaring $(F, G, \alpha, \beta) \sim (F', G', \alpha', \beta')$ exactly when there are natural isomorphisms $(\gamma : F \to F', \delta : G \to G')$ compatible with the pairs (α, β) and (α', β').

By following through the proof of Theorem 4.2, in this language it is asserted that there is a one-to-one bijection between quadruplets of the form (3) and equivalence classes of quadruplets of the form (4) under the equivalence relation just described.

It is clear that our Theorem 4.2 is the affine analogue of Theorem 1.6. Similarly, we note the next corollary which follows immediately from Theorem 4.2, and proves the affine version of Theorem 1.5.

Corollary 4.3.1. *Let R and S be rings. There is a canonical bijection between*

$$\mathrm{Spec}(R)(S) = \left\{ \textit{ring morphisms } R \to S \right\} \quad \textit{and} \quad \underline{\mathrm{Spec}}({}_R\mathrm{Mod})(S)$$

which extends to an isomorphism of categories.

5 The structure of the spectrum of $\mathrm{QCoh}(X)$

The goal of this section is to give an outline and explain the structure of the proof of Theorem 5.1 below. In particular, note that as a special case by the Yoneda lemma it follows immediately from this theorem that there is an isomorphism of functors

$$X(R) \cong \underline{\mathrm{Spec}}(\mathrm{QCoh}(X))(R)$$

natural in quasicompact quasiseparated schemes X and rings R.

Theorem 5.1 (Brandenburg–Chirvasitu, [2])**.** *Let X and Y be schemes with X quasicompact quasiseparated. Then taking the pullback defines a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{Sch}}(Y \to X) \cong \underline{\mathrm{Spec}}(\mathrm{QCoh}(X))(Y) = [\mathrm{QCoh}(X) \xrightarrow{\otimes,\ cocts.} \mathrm{QCoh}(Y)]/\sim.$$

Thus we begin by recalling the pullback construction.

Proposition 5.2. *Categories of quasicoherent sheaves over schemes are cocomplete and monoidal. Moreover, there is a functor (viewing* $\mathrm{Hom}_{\mathrm{Sch}}(Y \to X)$ *as a discrete category)*

$$(-)^* : \mathrm{Hom}_{\mathrm{Sch}}(Y \to X) \to [\mathrm{QCoh}(X) \xrightarrow{\otimes,\ cocts.} \mathrm{QCoh}(Y)]$$

called the pullback, which in particular factors through the cocontinuous monoidal functors.

Our key result Theorem 5.1 was first established by Brandenburg–Chirvasitu in [3], in some sense generalising[8] a result of Lurie [14] who proved an analogous statement with a certain condition called *tameness* imposed on the monoidal functors involved. Below we closely follow Brandenburg–Chirvasitu's original argument, delegating technical results to their original paper.

The proof of the is accomplished in three stages. First, the claim follows when $X = \operatorname{Spec} A$ for general categorical reasons. Next we handle the case when X is arbitrary quasicompact quasiseparated and Y is the spectrum of a local ring. Finally, we bootstrap to the case of general schemes Y.

We begin with some category-theoretic preliminaries. Recall that for an abelian category $\mathscr{A}$ the Yoneda embedding defines a map into the functor category $[\mathscr{A}^{\mathrm{op}} \to \mathrm{AbGrp}]$ (implicitly consisting of only the additive functors). The object $[\mathscr{A}^{\mathrm{op}} \to \mathrm{AbGrp}]$ is known as the *free cocompletion* of $\mathscr{A}$; one way to justify this is that it enjoys the following 2-universal property.

Theorem 5.3. *Every additive functor $F : \mathscr{A} \to \mathscr{B}$ into a cocomplete additive category $\mathscr{B}$ factors through the Yoneda embedding $Y : \mathscr{A} \to [\mathscr{A}^{\mathrm{op}} \to \mathrm{AbGrp}]$ up to unique natural isomorphism to yield a cocontinuous additive functor $\widetilde{F} : [\mathscr{A}^{\mathrm{op}} \to \mathrm{AbGrp}] \to \mathscr{B}$ called the* Yoneda extension.

In other words, for each cocomplete $\mathscr{B}$ there is a canonical equivalence of functor categories

$$[\mathscr{A} \xrightarrow{\text{cocts.}} \mathscr{B}] \simeq \Big[[\mathscr{A}^{\mathrm{op}} \to \mathrm{AbGrp}] \xrightarrow{\text{cocts.}} \mathscr{B} \Big].$$

Proof. This is Proposition 2.2.4 of [7]. □

Corollary 5.3.1. *If $\mathscr{A}$ and $\mathscr{B}$ are additive monoidal categories and $\mathscr{B}$ is cocomplete then the category $[\mathscr{A}^{\mathrm{op}} \to \mathrm{AbGrp}]$ has a canonical monoidal structure*[9]*, and there is a canonical equivalence of functor categories*

$$[\mathscr{A} \xrightarrow{\otimes,\ \text{cocts.}} \mathscr{B}] \simeq \Big[[\mathscr{A}^{\mathrm{op}} \to \mathrm{AbGrp}] \xrightarrow{\otimes,\ \text{cocts.}} \mathscr{B} \Big].$$

Proposition 5.4. *Let R be a ring and let $\mathbf{R}$ be the full subcategory of ${}_R\mathrm{Mod}$ containing just R.*[10] *The category ${}_R\mathrm{Mod}$ is the free cocompletion of $\mathbf{R}$.*

Proof. We need to check that ${}_R\mathrm{Mod} \simeq [\mathbf{R}^{\mathrm{op}} \to \mathrm{AbGrp}]$, but a functor $\mathbf{R}^{\mathrm{op}} \to \mathrm{AbGrp}$ just equips an abelian group $M \in \mathrm{AbGrp}$ with an R-action. □

Proposition 5.4 has the following immediate consequence.

Proposition 5.5. *Let R be a ring and let $\mathscr{B}$ be an additive monoidal category. Then there is a natural functor*

$$[{}_R\mathrm{Mod} \xrightarrow{\otimes,\ \text{cocts.}} \mathscr{B}] \to \mathrm{Hom}_{\mathbf{CAlg}}(R \to \mathrm{End}(\mathbb{1}_{\mathscr{B}})) \tag{5}$$

which restricts to $R \in {}_R\mathrm{Mod}$. When $\mathscr{B}$ is cocomplete this functor is part of an equivalence.

[8] Theorem 5.1 is not a generalisation of Lurie's result in a strict sense—Lurie in [14] proved a reconstruction theorem for categories of quasicoherent sheaves over arbitrary geometric stacks, showing that the so-called *tame* cocontinuous monoidal functors $\mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ recovered the collection of morphisms $Y \to X$. Brandenburg–Chirvasitu [3] lifted the tameness requirement in the case of schemes by imposing quasicompactness and quasiseparatedness conditions.

[9] This product is given by the so-called Day convolution, see [12, 6, 3].

[10] Since the only left R-module endomorphisms of R are left-multiplication by elements of R, we see that $\mathrm{Hom}_{\mathbf{R}}(R \to R) \cong R$. Thus $\mathbf{R}$ is just an encoding of R as a degenerate additive 1-category.

Proof. By Corollary 5.3.1 there is an equivalence

$$[\mathbf{R} \xrightarrow{\otimes,\text{ cocts.}} \mathscr{B}] \simeq [{}_R\mathrm{Mod} \xrightarrow{\otimes,\text{ cocts.}} \mathscr{B}].$$

The former object is canonically isomorphic to $\mathrm{Hom}_{\mathbf{CAlg}}(R \to \mathrm{End}(\mathbb{1}_{\mathscr{B}}))$, and tracing through the composite of these equivalences we find that the natural map (5) gives the functor in one direction. □

Proposition 5.6. *Theorem* 5.1 *holds in the special case of* $X = \operatorname{Spec} R$.

Proof. For any scheme Y, its morphisms into an affine scheme $X = \operatorname{Spec} A$ are parameterised by ordinary ring morphisms $A \to \Gamma(\mathscr{O}_Y)$ (this is Proposition 1.6.3 of [10]). But $\Gamma(\mathscr{O}_Y) = \mathrm{Hom}_{\mathrm{QCoh}(Y)}(\mathscr{O}_Y \to \mathscr{O}_Y)$ is exactly the set of endomorphisms of $\mathscr{O}_Y$ in $\mathrm{QCoh}(Y)$, so because $\mathrm{QCoh}(Y)$ is cocomplete Proposition 5.5 yields that there is an equivalence of categories

$$[{}_R\mathrm{Mod} \xrightarrow{\otimes,\text{ cocts.}} \mathrm{QCoh}(Y)] \simeq \mathrm{Hom}_{\mathrm{Sch}}(Y \to \operatorname{Spec} R).$$

Taking the quotient by isomorphisms we recover the statement of Theorem 5.1, as desired. (Observe that by tracing through the composites from right-to-left the equivalence is actually given by taking pullbacks.) □

With the fundamental base-case now established, from now on let X now be an arbitrary quasicompact quasiseparated scheme.

Definition 5.7. Let $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ be a monoidal functor, and let $j : U \hookrightarrow X$ be a quasicompact open immersion. We say that F is *U-local* if for every $\mathscr{F} \in \mathrm{QCoh}(X)$ the image $F(\mathscr{F}) \to F(j_* j^* \mathscr{F})$ of the canonical map[11] $\mathscr{F} \to j_* j^* \mathscr{F}$ is an isomorphism.

The use of the terminology "U-local" to describe this property of a functor $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ is justified since existence of a natural isomorphism $F(\mathscr{F}) \to F(j_* j^* \mathscr{F})$ implies that the action of F on some $\mathscr{F} \in \mathrm{QCoh}(X)$ is completely determined by the restriction of $\mathscr{F}$ to the subscheme U of X.

Remark 5.8. Observe that in the situation above the morphism $j^* j_* \mathscr{F} \to \mathscr{F}$ (actually, the counit of the ambient adjunction) is always an isomorphism. Moreover, [3] formulates U-locality as a special case of i-locality, where $(i^*, i_*, \eta, \varepsilon)$ is an adjunction with the counit ε an isomorphism. If i^* is a functor $\mathscr{C} \to \mathscr{D}$, then a functor F from $\mathscr{C}$ into another category $\mathscr{E}$ is i-local if the image of the unit η under F is also an isomorphism. Of course $(i^*, i_*) := (j^*, j_*)$ with $j : U \hookrightarrow X$ as above is an example.

Our previous results now allow us to establish the key tool used to reconstruct a morphism $Y \to X$ from functors $\mathrm{QCoh}(X) \xrightarrow{\otimes,\text{ cocts.}} \mathrm{QCoh}(Y)$, which reveals the purpose of the notion of U-locality.

Proposition 5.9. *Any cocontinuous monoidal functor* $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ *which is* $(\operatorname{Spec} A)$*-local arises as the pullback via a morphism* $Y \to X$.

Proof. The point is that we can form the composite

$$\mathrm{QCoh}(\operatorname{Spec} A) \xhookrightarrow{j_*} \mathrm{QCoh}(X) \xrightarrow{F} \mathrm{QCoh}(Y).$$

[11] This is the unit of the adjunction $j^* : \mathrm{QCoh}(X) \leftrightarrows \mathrm{QCoh}(U) : j_*$.

This is very close to the situation of Proposition 5.6, except for the fact that while j^* and j_* are adjoint and j^* is strong monoidal, as a result j_* is merely a lax monoidal functor. Fortunately, we are saved in this case by the fact that for formal category-theoretic reasons whenever F is strong monoidal and also U-local, actually $F \circ j_*$ is strong monoidal too (this is Proposition 2.3.6 of [3]).

We conclude that $F \circ j_*$ is induced by a morphism $f : Y \to \operatorname{Spec} A$, and post-composition with $\operatorname{Spec} A \hookrightarrow X$ yields the desired map $Y \to X$. □

The next step is to handle the case where $Y = \operatorname{Spec} R$ is affine, and at first in particular when R is a local ring. The utility of this assumption is established by the following pair of lemmas.

Lemma 5.10 (U-locality critereon for local rings)**.** *To determine that a cocontinuous monoidal functor* $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(\operatorname{Spec} R) \simeq {}_R\mathrm{Mod}$ *is U-local for $U \hookrightarrow X$ a quasicompact open immersion, it is sufficient to verify the following property:*

> For every closed subscheme $Z \hookrightarrow X$ for which $Z \times_X U = \emptyset$, the functor F maps the inclusion $\mathscr{I} \hookrightarrow \mathscr{O}_X$ of the quasicoherent ideal associated to Z to an isomorphism.

Proof. Brandenburg [3] calls the latter property *weak U-locality* of F. The proof is a technical result which we omit, but note that F maps into an honest category of R-modules, so the argument may largely take place there—an argument of a similar flavour appears in the proof of the next lemma. □

Lemma 5.11. *Any cocontinuous monoidal functor* $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(\operatorname{Spec} R) \simeq {}_R\mathrm{Mod}$ *into the category of modules over a local ring R is U-local for U some affine subscheme of X.*[12]

Proof. Since X is a quasicompact scheme it admits a finite open cover $\mathscr{U} = \{U_i = \operatorname{Spec} A_i \hookrightarrow X\}_{1 \le i \le n}$ by affine schemes. For a contradiction assume that X is not U_i-local for any $1 \le i \le n$. Now, each U_i is the complement of a closed subscheme Z_i of X with corresponding quasicoherent ideal $\mathscr{I}_i$ of $\mathscr{O}_X$. By the previous Lemma 5.10 we conclude that each image $F(\mathscr{I}_i) \hookrightarrow F(\mathscr{O}_X)$ is not an isomorphism for any i. We claim that if the inclusion $\mathscr{I}_i \hookrightarrow \mathscr{O}_X$ does not map to an isomorphism under F, then F does not even map it to a surjection. Assuming this for a moment, it follows that each $F(\mathscr{I}_i)$ factors through the inclusion of the maximal ideal $\mathfrak{m} \hookrightarrow A_i$. Therefore the image of the ideal sum $F(\sum_i \mathscr{I}_i)$ also factors through $\mathfrak{m}$, but since the U_i cover X this ideal sum must be all of $\mathscr{O}_X$. We conclude that F maps $\mathscr{O}_X$ into the maximal ideal of A, contradicting the fact that F is a monoidal functor. The claim follows.

It remains to prove the subclaim above. For this it sufficient to show that if $F(\mathscr{I}) \hookrightarrow F(\mathscr{O}_X)$ is a surjection then it is injective as well. This is a good example of reduction to a purely algebraic fact in the category of modules. Now, because F is a monoidal functor the multiplication in $\mathscr{I}$ yields a multiplication morphism $F(\mathscr{I}) \otimes F(\mathscr{I}) \to F(\mathscr{I})$, turning $F(\mathscr{I})$ into a nonunital R-algebra equipped with a surjective nonunital R-algebra morphism $t : F(\mathscr{I}) \to R$.

Since t is surjective we can find a preimage $u \in F(\mathscr{I})$ of the unit of R (i.e. satisfying $t(u) = 1$). Now suppose that $t(i) = 0$ for some $i \in F(\mathscr{I})$. But the multiplication map $\mathscr{I} \otimes \mathscr{I} \to \mathscr{I}$ in $\mathrm{QCoh}(X)$ factors through the unitor $\mathscr{O}_X \otimes \mathscr{I} \to \mathscr{I}$, so the same happens in ${}_R\mathrm{Mod}$ and this implies that that

$$i = i \lhd 1 = i \lhd t(u) = i \cdot u = t(i) \rhd u = 0.$$

[12] Our proof follows Lemmas 3.3.2 and 3.3.3 of [3].

We conclude that t is injective, as desired, and so this completes the proof. □

Thus we immediately obtain the following.

Proposition 5.12. *Theorem* 5.1 *holds in the special case of X quasicompact quasiseparated and $Y = \operatorname{Spec} R$ for R a local ring.*

Proof. By Lemma 5.11 any cocontinuous monoidal functor $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ is $(\operatorname{Spec} A)$-local for some open immersion $\operatorname{Spec} A \hookrightarrow X$, and so is in turn naturally isomorphic to a pullback f^* by some $f : Y \to X$ by Proposition 5.9. Conversely the pullback by any morphism $Y \to X$ is a cocontinuous monoidal functor. □

By bootstrapping from the local-ring case we also obtain the following.

Proposition 5.13. *Theorem* 5.1 *holds in the special case of X quasicompact quasiseparated and Y any affine scheme.*

Proof sketch. Let $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(\operatorname{Spec} R)$ be a cocomplete monoidal functor. We at least describe the construction for field points $x : \operatorname{Spec} K \to \operatorname{Spec} R$, leaving the rest to the proof of Proposition 3.4.1 of [3]. Indeed, any such map x factors through the localization of R at its prime ideal $\mathfrak{p} = \ker(x : R \to K)$. Post-composing F with the pullback along $\iota_{\mathfrak{p}} : \operatorname{Spec} R_{\mathfrak{p}} \to \operatorname{Spec} R$ yields a functor $\iota_{\mathfrak{p}} \circ F$ into $\mathrm{QCoh}(\operatorname{Spec} R_{\mathfrak{p}}) \simeq {}_{R_{\mathfrak{p}}}\mathrm{Mod}$ with $R_{\mathfrak{p}}$ a local ring. By Proposition 5.12 this composite is therefore naturally isomorphic to the pullback by a morphism $f_{\mathfrak{p}} : \operatorname{Spec} R_{\mathfrak{p}} \to X$. Pre-composing with $\pi : \operatorname{Spec} K \to \operatorname{Spec} R_{\mathfrak{p}}$ we obtain the desired K-point of X. The desired global morphism $\operatorname{Spec} R \to X$ is then determined by extending the assignment $x \mapsto \pi \circ f_{\mathfrak{p}}$ to maps from arbitrary affine schemes $\operatorname{Spec} S$. □

Noting a higher-categorical result for spectra of cocomplete monoidal categories, we are now able to establish the main theorem of this section.

Theorem 5.14. *For any cocomplete monoidal category $\mathscr{A}$, its categorical spectrum* $\mathbf{Spec}(\mathscr{A})$ *is an honest stack with respect to the Zariski topology on all schemes. In particular, its 1-categorical truncation* $\underline{\operatorname{Spec}}(\mathscr{A})$ *is a Zariski stack.*

Proof. This follows from Theorem 4.23 of [22]—in fact, as [3] points out, $\mathbf{Spec}(\mathscr{A})$ is even a stack in the stronger fpqc topology. □

Proof of Theorem 5.1. Of course, the proof proceeds by reduction to the previous cases. First, let $\mathscr{U} = \{j : U = \operatorname{Spec} A \hookrightarrow Y\}$ be an open cover by affine schemes. It is clear that the pullback defines a functor (in both X and Y)

$$\mathrm{Hom}_{\mathrm{Sch}}(Y \to X) \to \mathbf{Spec}(\mathrm{QCoh}(X))(Y).$$

It is not difficult[13] to check that this map is fully faithful, or equivalently, that existence of a monoidal natural transformation between pullback functors implies equality of the underlying morphisms. Moreover as a consequence of Theorem 5.14 invertibility can be checked locally in Sch, i.e. the claim of Theorem 5.1 is a Zariski-local one. Equivalently, a cocontinuous monoidal functor $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ is the pullback by a morphism $f : Y \to X$ if and only if each composite (with $j \in \mathscr{U}$)

$$\mathrm{QCoh}(X) \xrightarrow{F} \mathrm{QCoh}(Y) \xrightarrow{j^*} \mathrm{QCoh}(U)$$

[13]This follows from a strengthening of our Proposition 3.6 which uses quasiseparatedness of X in a critical way, see Proposition 3.1.1 of [3] for the details.

is induced by the pullback via a morphism $U \to X$. But now we can assume that Y is affine and directly appeal to Proposition 5.13, so we are done. $\square$

We conclude immediately that if $\mathrm{QCoh}(X)$ and $\mathrm{QCoh}(Y)$ are monoidally equivalent (and X is qcqs), then $X \cong Y$, and every such monoidal equivalence up to monoidal natural isomorphism arises from an ordinary morphism $f : Y \to X$. Therefore in particular, this proves Rosenberg's reconstruction theorem for monoidal equivalences $\mathrm{QCoh}(X) \overset{\otimes}{\simeq} \mathrm{QCoh}(Y)$.

6 Understanding arbitrary equivalences

The results of Brandenburg–Chirvasitu [2] thus establish the reconstruction theorem for monoidal functors $\mathrm{QCoh}(X) \overset{\otimes}{\simeq} \mathrm{QCoh}(Y)$, and it remains to handle the general case. Using the general Eilenberg–Watts theory of Nyman [16] we can extend the Eilenberg–Watts theorem to classes of categories of quasicoherent sheaves, and thereby continue to execute the general strategy which we applied to handle the affine case in Section 4. Of course, in order to generalize the Eilenberg–Watts theorem one must have a suitable notion of bimodules, and of the tensor product of quasicoherent sheaves over different schemes.

Thus we begin by explaining the basic construction. Let X and Y be suitably nice[14] schemes, and let $\mathscr{F}$ be a quasicoherent sheaf over the scheme $X \times Y$. Now fix $\mathscr{G} \in \mathrm{QCoh}(Y)$, and observe that the canonical projection $\pi_Y : X \times Y \to Y$ gives rise to a pullback functor $\pi_Y^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X \times Y)$. This means that we can take the tensor product of $\mathscr{F}$ with $\pi_Y^*\mathscr{G}$ in the category $\mathrm{QCoh}(X \times Y)$. The resulting object pushes forward under the projection $\pi_X : X \times Y \to X$ to yield an object $\pi_{X*}(\mathscr{F} \otimes_{\mathrm{QCoh}(X\times Y)} \pi_Y^*\mathscr{G})$ of $\mathrm{QCoh}(X)$. The end result is that we have essentially tensored a quasicoherent sheaf $\mathscr{G}$ over Y with a quasicompact "(X, Y)-bimodule" $\mathscr{F}$ to obtain a quasicoherent sheaf over X. This motivates the following definition.

Definition 6.1 (Generalized tensor product)**.** Let $\mathscr{F} \in \mathrm{QCoh}(X\times Y)$ and $\mathscr{G} \in \mathrm{QCoh}(Y)$. Whenever the pushforward π_{X*} exists we define the *generalized tensor product of $\mathscr{F}$ and $\mathscr{G}$* by

$$\mathscr{F} \otimes_{\mathrm{QCoh}(Y)} \mathscr{G} := \pi_{X*}(\mathscr{F} \otimes_{\mathrm{QCoh}(X\times Y)} \pi_Y^*\mathscr{G}) \in \mathrm{QCoh}(X).$$

The following definition[15] plays a key role in Nyman's development [16].

Definition 6.2. We say that a functor $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ is *totally global* if the composite $F \circ j_*$ is zero for every open immersion $j : U \hookrightarrow X$ from an affine scheme.

For our purposes, the main Theorem 1.4 of Nyman [16] specialises for X quasicompact separated and Y separated to give the following assertion.

Theorem 6.3 (Nyman, [16])**.** *Let $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ be a cocontinuous functor. There exists a canonical quasicoherent sheaf $\mathscr{W}_F \in \mathrm{QCoh}(Y \times X)$ and a canonical natural transformation*

$$\Gamma_F : F(-) \to \mathscr{W}_F \otimes_{\mathscr{O}_X} -$$

into the generalized tensor product with $\mathscr{W}_F$ called the "Eilenberg–Watts morphism". Moreover, if X is an affine scheme, or if F is exact, then Γ_F is a natural isomorphism.

[14] We will need that X is quasicompact, and that X and Y are both separated, and we will assume this for the remainder of this section.

[15] Basic consequences are established in Section 4 of [16].

Proof sketch. Nyman defines an *Eilenberg–Watts* functor

$$\mathscr{W}_{-} : \left[\mathrm{QCoh}(X) \xrightarrow{\text{cocts.}} \mathrm{QCoh}(Y) \right] \to \mathrm{QCoh}(Y \times X)$$

by first obtaining elements $\mathscr{W}_F^U \in \mathrm{QCoh}(Y \times U)$ for $U \hookrightarrow X$ in a finite affine open cover of X (thus we use quasicompactness of X), in turn obtained from the affine case established in [15] as Example 4.1. Separatedness of X is then used to glue the family $\{\mathscr{W}_F^U\}$ into the desired single quasicoherent sheaf over $Y \times X$ (see Subsection 5.2 of [16], in particular the separatedness hypothesis ensures that a family of pushforward maps exist).

The morphism Γ_F is then constructed by first being defined on flat objects of $\mathrm{QCoh}(X)$ and then extending—in particular, here it is used that every object of $\mathrm{QCoh}(X)$ is a quotient of a flat object whenever X is separated (e.g. by Lemma 1.1.4 of [17], see the Section 6 of [16]). We then show that Γ_F is compatible with restriction along affine open subschemes of Y, which in fact implies that $\ker \Gamma_F$ and $\operatorname{coker} \Gamma_F$ are totally global. Since totally global functors out of the category of quasicoherent sheaves over an affine scheme are zero (this is Nyman's Corollary 6.7 of [16]), the claim follows when $X = \operatorname{Spec} R$, and in fact another consequence is that the claim holds when F is exact as well. □

Wielding Theorem 6.3 in the case at hand, we finally obtain the following result in direct analogy with the affine case, and the original statement of Theorem 1.6.

Theorem 6.4 (Rosenberg's reconstruction theorem, scheme version)**.** *Fix schemes X and Y. If X is quasicompact separated and Y is separated then there is a canonical bijection between the sets*

$$\left\{ (f : Y \to X, [L] \in K_0(\mathrm{QCoh}(Y))) \mid f \textit{ is an isomorphism and } L \textit{ is a line bundle} \right\}$$

and

$$\underline{\mathrm{Spec}}(\mathrm{QCoh}(X))(Y) = \left\{ F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y) \mid F \textit{ is an equivalence} \right\} / \sim$$

where the relation $\sim$ is the quotient by natural isomorphism.

In particular, the existence of any equivalence $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ at all implies the existence of an isomorphism $f : Y \to X$.

Proof. By Theorem 6.3 an equivalence $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ is naturally isomorphic to tensoring over $\mathrm{QCoh}(X)$ with some $\mathscr{W}_F \in \mathrm{QCoh}(Y \times X)$. On the other hand a quasi-inverse $G : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ tensors with some $\mathscr{W}_G \in \mathrm{QCoh}(X \times Y)$. As in the affine case, the natural isomorphisms $FG \to \mathrm{id}_{\mathrm{QCoh}(Y)}$ and $GF \to \mathrm{id}_{\mathrm{QCoh}(X)}$ imply that tensoring with $\pi_{Y*}\mathscr{W}_F \in \mathrm{QCoh}(Y)$ is invertible, hence determines an autoequivalence E of $\mathrm{QCoh}(Y)$. Thus we obtain a composite of equivalences $E \circ F$ which preserves the tensor unit of $\mathrm{QCoh}(X)$. As before $E \circ F$ upgrades to a cocontinuous and now also monoidal functor, and so therefore by appeal to the main Theorem 5.1 of Section 5, the functor F is realized as the pullback by a morphism $f : Y \to X$ of ordinary schemes. The same argument can be employed beginning with G instead of F, and functoriality of the constructions involved implies that we obtain a double-sided inverse g of f—hence f is an isomorphism. Conversely each pair $(f, [L])$ gives rise to an equivalence $\mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ as in the affine case, so this completes the proof. □

7 The punchline: noncommutative schemes

The main theorem of Brandenburg–Chirvasitu in [3] asserts that, when we restrict to quasicompact quasiseparated schemes, the collection of morphisms of schemes $Y \to X$ is recovered from the collection of morphisms from $\mathrm{QCoh}(X)$ to $\mathrm{QCoh}(Y)$, for a suitable notion of the latter kind of morphism. One way to interpret this result is that it marks the starting point of *noncommutative algebraic geometry*.[16]

Indeed, there is a general program for producing a noncommutative analogue of a more classical "commutative" construction; we show that a certain category $\mathscr{C}$ consisting of the original "commutative" objects embeds into a larger category $\mathscr{D}$, whereby objects of $\mathscr{C}$ can be viewed as objects of $\mathscr{D}$ satisfying a particular condition. To obtain analogous "noncommutative" objects, we simply drop this additional condition and allow arbitrary objects of $\mathscr{D}$.

In our case then, forming the category of quasicoherent sheaves is a 2-functor QCoh from Sch into the 2-category of cocomplete abelian monoidal categories, and Brandenburg–Chirvasitu's Theorem 5.1 asserts that this embedding is fully faithful in the appropriate 2-categorical sense (at least on a suitable full subcategory of Sch). Explicitly, a reasonable guess for the right notion of a noncommutative scheme becomes (approximately) a cocomplete abelian monoidal category. Indeed [5] introduces the category of *commutative 2-rings*, with objects symmetric presentable monoidal categories for which the tensor product distributes over colimits—and of which our categories of quasicoherent sheaves are examples. One can then begin to work with "higher algebraic geometry" using the category of 2-affine schemes, opposite to the category of commutative 2-rings. From this viewpoint our usual schemes and all of 1-algebraic geometry embeds via QCoh into the opposite category of 2-affine schemes. Thus another interpretation of Brandenburg–Chirvasitu's main theorem is that "all of 1-algebraic geometry is 2-affine" [3].

On the other hand, the spectrum $\underline{\mathrm{Spec}}$ of a cocomplete abelian monoidal category is only one of several possible methods of "constructing" a noncommutative scheme. Rosenberg (e.g. in [20]) has developed a number of other notions of spectra including of triangulated categories, or of his "right exact" categories. There is also the entire subject of derived noncommutative geometry, where one represents an ordinary scheme X as its derived category of quasicoherent sheaves, or an A_∞-category enrichment thereof.[17]

Versions of Rosenberg's reconstruction theorem also extend past the boundaries of ordinary schemes. Calabrese–Groechenig in [4] extended the reconstruction theorem to quasicompact algebraic spaces, and gave (in comparison to Rosenberg and Brandenburg) a proof along entirely different lines. Schäppi [21] has proved the direct analogue of Theorem 5.1 for so-called Adams stacks. If we restrict to equivalences which respect the monoidal structure, then the reconstruction theorem of Lurie [14] is able to recover arbitrary geometric stacks. The list certainly goes on, and this essay cannot, but hopefully some impression of the breadth of available possibilities has been left; and we finish having at least made a start.

[16] The 1995 book [18] of Rosenberg, and the 1994 paper [1] of Artin and Zhang, are famous examples of definitions of noncommutative analogues in algebraic geometry which utilize abelian categories [2].

[17] One of the primary motivations of this sort of construction is homological mirror symmetry, see [13].

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