The Stone–von Neumann–Mackey theorem
Quantum mechanics in functional analysis

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In this essay we explain the proof of a pair of results in functional analysis arising from the historical development of quantum mechanics: these are *Stone's theorem* on strongly continuous unitary groups and the *Stone–von Neumann* theorem. We do this from the perspective of an arbitrary locally compact abelian group G, though these results were originally stated for $G = \mathbf{R}$ or $G = \mathbf{R}^n$. The first of them, Stone's theorem, characterizes all strongly continuous one-parameter unitary groups acting on a Hilbert space $\mathcal{H}$. The second, the Stone–von Neumann theorem, is a successor in spirit and characterizes pairs of (strongly continuous) unitary representations of $\mathbf{R}^n$ on $\mathcal{H}$ satisfying the so-called *exponentiated canonical commutation relations* of quantum mechanics.

In order to state and prove natural analogues of these results for arbitrary locally compact abelian groups, we combine powerful structural theorems from functional/harmonic analysis together with results from representation theory and C*-algebra theory. In particular we adopt the modern *-algebraic perspective on the proof of the Stone–von Neumann theorem, in which *Green's imprimitivity theorem* plays the key role. Green's imprimitivity theorem is in some sense an abstract formulation of the Stone–von Neumann theorem which makes sense even for nonabelian groups. Finally in the context of this framework we are able to outline further generalizations, both from a physical perspective (in the direction of *supersymmetry*), and from a mathematical perspective (the case of nonabelian groups).

Contents

1 Basic representation theory: the group algebra

Throughout we fix a locally compact (Hausdorff) abelian group G. We label the identity by $1 \in G$, and always use the convention of a multiplicative group operation. If $S \subset X$ is a measurable subset of a measure space X then χ_S denotes the characteristic function for the set S. We use $C(Y \to Z)$ to denote the set of continuous maps $Y \to Z$, and write $C(Y) := C(Y \to \mathbf{C})$. We also let $\mathcal{B}(\mathcal{H})$ denote the set of bounded linear operators on a Hilbert space $\mathcal{H}$. We use $\sigma(A)$ to denote the spectrum of a linear operator $A : D_A \to \mathcal{H}$. All of our associative or Banach algebras will be nonunital unless otherwise stated. We assume basic results in functional analysis on Hilbert spaces, in particular the (various forms of the) spectral theorem for self-adjoint linear operators, and existence of the associated Borel functional calculus.

We begin by recalling some basic results in the theory of locally compact groups—especially regarding their representation theory—which we will require in the sequel.

Theorem 1.1 (Haar [4]). *There exists a countably additive nontrivial regular Borel measure μ on G which is finite on every compact subset of G and is invariant in the sense that for all $g \in G$ and Borel subsets $S \subseteq G$ we have $\mu(gS) = \mu(S)$. Moreover, μ is unique up to multiplication by a positive constant.*

Henceforth we fix a particular choice of measure μ satisfying the hypotheses of Theorem 1.1, which we call the *Haar measure*[1] on G. We are now able to define the group algebra of the locally compact group G, generalising the ordinary group algebra $\mathbf{C}[G]$ of a finite group.

Definition 1.2. The *group algebra of G* is $L^1(G) := L^1(G, \mu)$, the collection of absolutely integrable complex-valued functions on G. In fact $L^1(G)$ is naturally a (nonunital) associative algebra over the complex numbers $\mathbf{C}$ (scalar multiplication is done pointwise) with respect to the convolution operation

$$(f_1 * f_2)(g) = \int_G f_1(g') f_2(g'^{-1} g) \,\mathrm{d}\mu(g').$$

In particular by the inequality $\|f_1 * f_2\|_{L^1(G)} \leq \|f_1\|_{L^1(G)} \|f_2\|_{L^1(G)}$ for L^p-spaces we conclude that $f_1 * f_2 \in L^1(G)$ and $L^1(G)$ is a Banach algebra. Because G is abelian we see readily that this multiplication is commutative. If $\mu(G) < \infty$ then constant functions lie in $L^1(G)$. (Note that, though $L^1(G)$ might contain the constants the algebra $L^1(G)$ still need not be unital.)

There is also an involution on $L^1(G)$ defined by

$$f^*(x) = \overline{f(x^{-1})},$$

and it is easy to see that this operation turns $L^1(G)$ into a complex commutative Banach *-algebra.

Remark 1.3. When G is a finite (abelian) group the Haar measure μ on G is just the counting measure, and $L^1(G) \cong \mathbf{C}[G]$ is an isomorphism of Banach *-algebras in the natural way.

Of chief importance amongst the kinds objects we will consider is the notion of a unitary representation of the locally compact abelian group G, which we now define.

Definition 1.4. A *unitary representation ρ of G on a Hilbert space $\mathcal{H}$* is a homomorphism from G to the monoid of bounded linear operators $\mathcal{B}(\mathcal{H})$ on $\mathcal{H}$, such that:

1. the image of ρ is contained in the unitary operators, and
2. the map ρ is continuous with respect to the strong operator topology on $\mathcal{B}(\mathcal{H})$. This is the coarsest topology on $\mathcal{B}(\mathcal{H})$ such that for each fixed $x \in \mathcal{H}$ the evaluation map $\mathrm{ev}_x : \mathcal{B}(\mathcal{H}) \to \mathcal{H}$ (defined by $\mathrm{ev}_x(T) := T(x)$) is continuous.

Let $\rho : G \to \mathcal{B}(\mathcal{H})$ and $\rho' : G \to \mathcal{B}(\mathcal{H}')$ be a pair of unitary representations. We say that bounded linear operator $A : \mathcal{H} \to \mathcal{H}'$ is a *morphism $\rho \to \rho'$ of unitary representations* if $A \circ \rho(g) = \rho'(g) \circ A$ for all $g \in G$. The representations ρ and ρ' are *(unitarily) isomorphic* if there exists a morphism $U : \rho \to \rho'$ which is itself unitary.

[1]When G is not abelian we obtain essentially unique left- and right-Haar measures μ_l and μ_r in the same way, depending on whether we demand that μ-volumes are preserved by the G-action on Borel subsets of G on the left or right side, respectively.

A unitary representation $\rho : G \to \mathcal{B}(\mathcal{H})$ is *irreducible* if $\mathcal{H}$ admits no nontrivial ρ-invariant subspaces, i.e. there are no proper nonzero closed subspaces of $\mathcal{H}$ invariant under all of the operators in the set $\rho(G)$.

Example 1.5. We always have the *left regular representation* of G on $L^2(G) := L^2(G, \mu)$ defined by letting $g \in G$ act on $f \in L^2(G)$ by

$$(g \cdot f)(g') := f(g^{-1}g').$$

This action is unitary since

$$\|g \cdot f\|_{L^2(G)}^2 = \int_G |f(g^{-1}g')|^2 \, \mathrm{d}\mu(g') = \int_G |f(g')|^2 \, \mathrm{d}\mu(g') = \|f\|_{L^2(G)}^2$$

by ordinary change-of-variables (recall that G acts on itself regularly and μ-volumes are invariant under G). More generally, it will be useful to define an action of G on the group algebra $L^1(G)$ by the same formula, which we also refer to as the left regular representation (now of G on $L^1(G)$).

Lemma 1.6 (Schur [4, 6])**.** *Fix a unitary representation $\rho : G \to \mathcal{B}(\mathcal{H})$. The following are equivalent:*

(1) the representation ρ is irreducible,

(2) every morphism $A : \rho \to \rho$ which is an orthogonal projection is zero or the identity, and

(3) every morphism $A : \rho \to \rho$ is multiplication by a constant.

Proof. (1) $\implies$ (2) : Assume that ρ is irreducible and that a morphism $A : \rho \to \rho$ is an orthogonal projection. Then in particular if $x \in \ker A$ for each $g \in G$ we have $A\rho(g)x = \rho(g)Ax = 0$, showing that $\ker A$ is a ρ-invariant subspace of $\mathcal{H}$. Hence $\ker A = \{\mathbf{0}\}$ or $\ker A = \mathcal{H}$, as desired.

(2) $\implies$ (3) : We use the following trick exploiting the existence of the Borel functional calculus for self-adjoint operators[2]; fix a morphism $A : \rho \to \rho$. Both of the operators $A_1 = A + A^*$ and $A_2 = i(A - A^*)$ on $\mathcal{H}$ are automatically also morphisms $\rho \to \rho$. It suffices to show that A_1 and A_2 are both constant since then A is therefore constant as well; thus assume for a contradiction that A_i is nonconstant for $i \in \{1, 2\}$. By construction the operator A_i is self-adjoint, and the spectrum $\sigma(A_i)$ cannot be a singleton by the nonconstantcy hypothesis, so we may find a nonempty proper Borel subset $S \subsetneq \sigma(A_i)$. The functional calculus turns the characteristic function χ_S into an orthogonal projection $\chi_S(A_i)$ on $\mathcal{H}$, and since $\rho(g)$ commutes with A_i for all $g \in G$, we conclude that $\chi_S(A_i)$ is a morphism $\rho \to \rho$ as well. Hence by hypothesis $\chi_S(A_i)$ is either the identity on $\mathcal{H}$ or the zero map, and both are impossible by construction (in particular, by elementary properties of the Borel functional calculus).

(3) $\implies$ (1) : Let $V \subseteq \mathcal{H}$ be a (closed) ρ-invariant subspace, and let $A : \mathcal{H} \to V$ be the associated orthogonal projection. Fix $g \in G$. Then if $x \in V$ we have $A\rho(g)x = \rho(g)x = \rho(g)Ax$ since $\rho(g)x \in V$ by ρ-invariance. On the other hand if $x \in V^\perp$ then for arbitrary $y \in V$ we have $(\rho(g)x, y) = (x, \rho(g^{-1})y) = 0$ by ρ-invariance of V again, showing that $\rho(g)x \in V^\perp$. Then $A\rho(g)x = 0 = \rho(g)Ax$ because A is an orthogonal projection, also as desired. Therefore A is a morphism $\rho \to \rho$, so must be multiplication by a constant. Hence A is either zero or the identity on $\mathcal{H}$, and we conclude that V is either zero or all of $\mathcal{H}$. □

Both Stone's theorem and the Stone–von Neumann theorem make statements about the isomorphism classes of unitary representations of G subject to various conditions. For the moment we restrict our attention to only the irreducible unitary representations.[3]

Definition 1.7. The *dual* $G^\vee$ of a locally compact abelian group G is the set of irreducible unitary representations of G, modulo unitary isomorphism of representations.

[2]The idea shown here appears as part of the proof of Lemma 3.5 of [6], or Theorem 5.1.6 of [4].

[3]In attempting to extend the analogy with the case of finite groups, one might ask at this point whether we have an analogue of Maschke's theorem in the locally compact group case (giving so-called "complete reducibility" of representations of finite groups as direct sums of irreducible representations). Though we will not need this fact, it is true that we have such a direct sum decomposition whenever G is compact (see for instance Theorem 15.1.3 of [5]). In the general case any representation may instead be exhibited (by von Neumann's construction [20, 6]) as a *direct integral* of irreducible representations. Several of our results below are related to this perspective (in particular Theorem 2.9), and may be viewed as a special case of the direct integral technology.

The notion of a unitary representation and the dual (of Definition 1.7) both make sense even when G is not abelian. However, in the abelian case we enjoy the following result.[4]

Lemma 1.8. *Every irreducible unitary representation $\rho : G \to \mathcal{B}(\mathcal{H})$ is 1-dimensional, and therefore canonically corresponds to a continuous group homomorphism $\widetilde{\rho} : G \to \mathbf{T}$, i.e. which maps G into the unit circle $\mathbf{T}$. Conversely, each continuous group homomorphism $G \to \mathbf{T}$ corresponds to an irreducible unitary representation of G and, modulo unitary isomorphism of unitary representations, these constructions are in mutual bijection.*

Proof. Fix an irreducible unitary representation $\rho : G \to \mathcal{B}(\mathcal{H})$ and let $g \in G$ be arbitrary. Then for any $g' \in G$ we have $\rho(g) \cdot \rho(g') = \rho(g \cdot g') = \rho(g' \cdot g) = \rho(g') \cdot \rho(g)$ by abelianness. Since $\rho(g)$ is unitary, the map $\rho(g) : \mathcal{H} \to \mathcal{H}$ is itself a unitary isomorphism $\rho \to \rho$.

At this point we use Schur's lemma for unitary representations (Lemma 1.6) to conclude that $\rho(g) = \lambda_g I$ for some $\lambda_g \in \mathbf{T}$. In particular, every subspace of $\mathcal{H}$ is invariant under $\rho(g)$, and since $g \in G$ was arbitrary, this means that every subspace of $\mathcal{H}$ is ρ-invariant. Since ρ is assumed irreducible, we conclude that the only possibility is that $\dim \mathcal{H} = 1$, as desired. Indeed, in this case we may define a map $\widetilde{\rho} : G \to \mathbf{T}$ by setting $\widetilde{\rho}(g) = \lambda_g$, and observe that since ρ is strongly continuous we have that $\widetilde{\rho}$ is continuous (indeed, in this case the norm and strong topologies are equivalent), and similarly $\widetilde{\rho}$ is also a group homomorphism because $\widetilde{\rho}(g_1 g_2)I = \rho(g_1 g_2) = \rho(g_1)\rho(g_2) = \widetilde{\rho}(g_1)\widetilde{\rho}(g_2)I$. In much the same way, letting a continuous group homomorphism $\sigma : G \to \mathbf{T}$ be given we obtain a unitary operator $\sigma(g)I \in \mathcal{B}(\mathbf{C})$ for all $g \in G$, and further the map $g \mapsto \sigma(g)I$ is then a strongly continuous unitary representation of G on $\mathbf{C}$. Obviously $\widetilde{\rho}$ determines the unitary isomorphism class of the representation ρ, so we conclude that these two constructions are obviously mutually inverse, and this completes the proof. □

Through Lemma 1.8, we may equivalently view $G^\vee$ as a subset of $C(G \to \mathbf{T})$, the set of all continuous maps from G into the circle group. This latter object is naturally equipped with the compact-open topology, and thus $G^\vee$ becomes a topological space by inheriting the subspace topology. Moreover $G^\vee$ becomes a group by performing multiplication pointwise in $\mathbf{T}$. In fact, these structures enjoy a fortunate compatibility.

Proposition 1.9. *The topology on $G^\vee$ is Hausdorff and the natural (pointwise) multiplication and inversion operations in $G^\vee$ are continuous with respect to this topology.*

Proof. This follows directly from recognizing the compact-open topology as the topology of uniform convergence on compact sets. □

The natural question is now whether $G^\vee$ is locally compact, and thus whether $G^\vee$ is itself a locally compact abelian group. Theorem 2.8 of the next section will decide this question in the affirmative. As a consequence, via the perspective of Lemma 1.8 we may define a group homomorphism $\mathbf{R}^n \to (\mathbf{R}^n)^\vee$ simply by $x \mapsto (y \mapsto e^{2\pi i x \cdot y})$. In fact we have the following.[5]

Lemma 1.10. *The map $\mathbf{R}^n \to (\mathbf{R}^n)^\vee$ defined by $x \mapsto (y \mapsto e^{2\pi i x \cdot y})$ is a homeomorphism and a group isomorphism.*

Proof. The homomorphism property and injectivity of the map are both clear, so we turn to verifying surjectivity. Thus fix $\psi \in (\mathbf{R}^n)^\vee$. Since ψ is a homomorphism we may assume $n = 1$. In fact multiplicativity of ψ implies that ψ is differentiable at 0; since ψ is continuous and $\psi(0) = 1$ there exists $\varepsilon > 0$ such that the integral $C = \int_0^\varepsilon \psi(t)\,\mathrm{d}t$ is nonzero. Then for all $y \in \mathbf{R}$ we have

$$\psi(y) = \frac{1}{C}\int_0^\varepsilon \psi(y)\psi(t)\,\mathrm{d}t = \frac{1}{C}\int_0^\varepsilon \psi(y+t)\,\mathrm{d}t = \frac{1}{C}\left(\int_0^{y+\varepsilon} \psi(t)\,\mathrm{d}t - \int_0^{y} \psi(t)\,\mathrm{d}t\right),$$

and hence $\psi'(y) = \frac{1}{C}(\psi(y+\varepsilon) - \psi(y)) = \frac{\psi(\varepsilon)-1}{C}\psi(y)$. The unique solution to this equation satisfying $\psi(0) = 1$ is $\psi(y) = e^{2\pi i \frac{\psi(\varepsilon)-1}{2\pi i C} x}$. Finally, recalling that the topology on $(\mathbf{R}^n)^\vee$ is just given by uniform convergence on compact subsets we immediately recognize the map $\mathbf{R}^n \to (\mathbf{R}^n)^\vee$ as a homeomorphism, so this completes the proof. □

[4]We may view this lemma as a generalization of the 1-dimensionality of irreducible representations of finite abelian groups.

[5]This trick exploiting integration appears, for example, as Theorem 9.11 of [3] or Theorem 4.5 of [6].

It will be of great utility for us to recast continuous (with respect to the strong topology on the target) homomorphisms $G \to \mathcal{B}(\mathcal{H})$ as maps of a more purely algebraic flavour. By analogy, consider the correspondence between representations of a finite group G on a finite dimensional vector space V and morphisms from the group algebra $\mathbf{C}[G]$ into the endomorphism algebra $\mathrm{End}(V)$). Thus we make the following definition.

Definition 1.11. A **-representation* of a Banach *-algebra A on a Hilbert space $\mathcal{H}$ is a continuous *-algebra homomorphism into the Banach *-algebra $\mathcal{B}(\mathcal{H})$. A *-representation $\eta : A \to \mathcal{B}(\mathcal{H})$ is *nondegenerate* if $\eta(A)\mathcal{H}$ is dense in $\mathcal{H}$.

The next pair of results make precise the analogy between $\mathbf{C}[G]$ and $\mathrm{End}(V)$, and $L^1(G)$ and $\mathcal{B}(\mathcal{H})$. Note the presence of the nondegeneracy condition, which we will use in the next section where we consider generalisations of the spectral theorem for Hilbert spaces.

Proposition 1.12 ([5, 6])**.** *Let ρ be a unitary representation of G on a Hilbert space $\mathcal{H}$. Then the map $\Gamma(\rho) : L^1(G) \to \mathcal{B}(\mathcal{H})$ defined by the Bochner integral*[6]

$$\Gamma(\rho) : f \mapsto \int_G f(g)\rho(g)\,\mathrm{d}\mu(g), \tag{1}$$

*i.e. integration against ρ, is a (bounded) nondegenerate *-representation of $L^1(G)$.*

Proof. Fix a unitary representation ρ of G on $\mathcal{H}$. The integral (1) giving $\Gamma(\rho)$ always exists by Bochner's criterion for integrability[7], since for each fixed $f \in L^1(G)$ we have $\|f(g)\rho(g)\| \le \|f(g)\|$ for all $g \in G$ because ρ is a unitary representation. Linearity of the assignment $f \mapsto \Gamma(\rho)(f)$ is then automatically linear, and this map is also bounded by monotonicity of the Bochner integral; we have the inequality

$$\|\Gamma(\rho)(f)\|_{\mathcal{B}(\mathcal{H})} = \left\| \int_G f(g)\rho(g)\,\mathrm{d}\mu(g) \right\|_{\mathcal{B}(\mathcal{H})} \le \int_G \|f(g)\rho(g)\|_{\mathcal{B}(\mathcal{H})}\,\mathrm{d}\mu(g) = \|f\|_{L^1(G)}.$$

With the bounded linear map $\Gamma(\rho) : L^1(G) \to \mathcal{B}(\mathcal{H})$ now in hand, we may now directly compute for each $f_1, f_2 \in L^1(G)$ and $x, y \in \mathcal{H}$ that (by Fubini's theorem together with elementary properties of the Bochner integral)

$$\begin{aligned}
(\Gamma(\rho)(f_1 * f_2)x, y) &= \int_G (f_1 * f_2)(g_1)(\rho(g_1)x, y)\,\mathrm{d}\mu(g_1) \\
&= \int_G \int_G f_1(g_2^{-1}g_1)f_2(g_2)\,\mathrm{d}\mu(g_2)(\rho(g_1)x, y)\,\mathrm{d}\mu(g_1) \\
&= \int_G \int_G f_1(g_1)f_2(g_2)(\rho(g_1)\rho(g_2)x, y)\,\mathrm{d}\mu(g_1)\,\mathrm{d}\mu(g_2) \\
&= \int_G f_1(g_1) \int_G f_2(g_2)(\rho(g_2)x, \rho(g_1^{-1})y)\,\mathrm{d}\mu(g_2)\,\mathrm{d}\mu(g_1) \\
&= \int_G f_1(g_1)(\Gamma(\rho)(f_2)(x), \rho(g_1^{-1})y)\,\mathrm{d}\mu(g_1) \\
&= \int_G f_1(g_1)(\rho(g_1)\Gamma(\rho)(f_2)(x), y)\,\mathrm{d}\mu(g_1) \\
&= \Gamma(\rho)(f_1)\Gamma(\rho)(f_2)(x), y),
\end{aligned}$$

[6] In order to avoid the Bochner integral machinery entirely we may view this definition instead as simply specifying for each $x, y \in \mathcal{H}$ and $f \in L^1(G)$ that there is an operator $\Gamma(\rho)(f)$ satisfying $(\Gamma(\rho)(f)(x), y) = \int_G f(g)(\rho(g)x, y)\,\mathrm{d}\mu(g)$—this is done for example in [4]. To show that this data actually then assembles into a bounded linear map on $\mathcal{H}$ we may then proceed as in Proposition C.3.1 of [4]; first show that the map $y \mapsto \overline{(\Gamma(\rho)(f)(x), y)}$ on $\mathcal{H}$ is a bounded linear functional, then appeal to Riesz representation theorem to obtain a vector $z_x \in \mathcal{H}$ such that $(\Gamma(\rho)(f)(x), y) = (z_x, y)$ for all $y \in \mathcal{H}$, and finally show that the assignment $x \mapsto z_x$ is itself a bounded linear map.

[7] For a reference see for instance Appendix 3 of [6].

showing that $\Gamma(\rho)(f_1 * f_2) = \Gamma(\rho)(f_1) \circ \Gamma(\rho)(f_2)$. Similarly we also have

$$\begin{aligned}(\Gamma(\rho)(f^*)x, y) &= \int_G f^*(g)(\rho(g)x, y)\,\mathrm{d}\mu(g)\\ &= \int_G \overline{f(g^{-1})}(\rho(g)x, y)\,\mathrm{d}\mu(g)\\ &= \overline{\int_G f(g)\overline{(\rho(g^{-1})x, y)}\,\mathrm{d}\mu(g)}\\ &= \overline{\int_G f(g)(\rho(g^{-1})^{-1}y, x)\,\mathrm{d}\mu(g)}\\ &= \overline{(\Gamma(\rho)(f)y, x)}\\ &= (x, \Gamma(\rho)(f)y),\end{aligned}$$

and therefore $\Gamma(\rho)(f^*) = \Gamma(\rho)(f)^*$. It follows immediately that $\Gamma(\rho)$ is a $*$-homomorphism.

It just remains to show that $\Gamma(\rho)$ is nondegenerate. Thus fix $x \in \mathcal{H}$ and $\varepsilon > 0$, and consider the continuous map $\delta : G \to \mathbf{R}$ defined by $\delta : g \mapsto \|\rho(g)x - x\|_{\mathcal{H}}$. Because $\delta(1) = 0$, the (open) preimage $U_\varepsilon := \delta^{-1}([0, \varepsilon))$ is nonempty. By inner regularity of the Haar measure we may find a compact subset $C_\varepsilon \subseteq U_\varepsilon$ with finite and positive measure. Then we have (again using monotonicity of the Bochner integral, now for an $\mathcal{H}$-valued integral only)

$$\begin{aligned}\left\|\Gamma\left(\frac{1}{\mu(C_\varepsilon)}\chi_{C_\varepsilon}\right)(x) - x\right\|_{\mathcal{H}} &= \left\|\int_G \frac{1}{\mu(C_\varepsilon)}\chi_{C_\varepsilon}(g)\rho(g)x\,\mathrm{d}\mu(g) - x\right\|_{\mathcal{H}}\\ &= \left\|\frac{1}{\mu(C_\varepsilon)}\int_G \chi_{C_\varepsilon}(g)(\rho(g)x - x)\,\mathrm{d}\mu(g)\right\|_{\mathcal{H}}\\ &\le \frac{1}{\mu(C_\varepsilon)}\int_{C_\varepsilon} \|\rho(g)x - x\|_{\mathcal{H}}\,\mathrm{d}\mu(g)\\ &\le \varepsilon.\end{aligned}$$

Since $x \in \mathcal{H}$ and $\varepsilon > 0$ were chosen arbitrarily, this establishes that $\Gamma(\rho)(L^1(G))\mathcal{H}$ is dense in $\mathcal{H}$, and hence completes the proof. □

Theorem 1.13. *The map Γ of Proposition* 1.12*, from unitary representations of G to nondegenerate *-representations of $L^1(G)$, is a bijection. Moreover, Γ respects irreducibility of representations.*

Proof idea. We will not need this result in the proof of Stone's theorem—though we would be remiss to overlook it—so the proof is omitted. One reference is [6], in which this result appears as Theorem 3.11; the key idea is to recover $\rho(g)$ from the value of $\Gamma(\rho)$ by evaluating $\Gamma(\rho)(f_n)$ where (f_n) is a sequence in $L^1(G)$ of nonnegative unit norm functions which become localized in an arbitrarily small neighbourhood of $g \in G$.[8] □

2 The spectral theorem and a consequence: Stone's theorem

With the necessary tools from representation theory now in hand, we begin by recalling a key result in functional analysis and C*-algebras: the spectral theorem. We will be able to deduce Stone's theorem by carefully characterizing the so-called "spectrum" of the group algebra $L^1(G)$.

Definition 2.1. Let A be a commutative Banach algebra (not necessarily unital). Then the *spectrum* $\Xi(A)$ *of* A is the set of nonzero linear functionals φ on A which are also multiplicative, i.e. for which $\varphi(a_1a_2) = \varphi(a_1)\varphi(a_2)$ for all $a_1, a_2 \in A$. The set $\Xi(A)$ becomes a topological space when equipped with the topology of pointwise convergence.

[8]The fundamental property at play here is that, though $L^1(G)$ is usually not unital, it does possess a net which is a so-called *bounded approximate identity*.

The spectrum of a commutative Banach algebra provides a natural setting for a generalization of the spectral theorem to take place. The key ingredient is the following.

Definition 2.2. Let A be a commutative Banach algebra. The *Gelfand transform* is the map $A \longrightarrow C(\Xi(A))$ defined for each $a \in A$ by evaluation-at-a, i.e. $\widehat{a}(\varphi) := \varphi(a)$.

Theorem 2.3 (C*-spectral theorem, projection version)**.** *Let A be a commutative unital C*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Then there exists a unique regular $\mathcal{H}$-projection valued measure E on $\Xi(A)$ such that for each $a \in A$ we have*

$$a = \int_{\Xi(A)} \widehat{a}(\psi)\,\mathrm{d}E(\psi). \tag{2}$$

In particular this means that for each $x, y \in \mathcal{H}$ there is a regular complex Borel measure $E_{x,y}(\lambda)$ on $\Xi(A)$ such that

$$(ax, y) = \int_{\Xi(A)} \widehat{a}(\psi)\,\mathrm{d}E_{x,y}(\psi).$$

We will actually require the following generalisation of the spectral theorem to commutative Banach *-algebras.

Theorem 2.4 (Banach *-algebra spectral theorem, projection version)**.** *Let A be a commutative Banach *-algebra and let $\eta : A \longrightarrow \mathcal{B}(\mathcal{H})$ be a nondegenerate *-representation thereof. Then there again exists a unique regular $\mathcal{H}$-projection valued measure E on $\Xi(A)$ such that for all $a \in A$ we have that the generalization*

$$\eta(a) = \int_{\Xi(A)} \widehat{a}(\psi)\,\mathrm{d}E(\psi).$$

of (2) *holds.*

Proof. This is Theorem 1.54 of [6], generalizing the C*-version which appears as Theorem 1.44. In particular note that the unitality requirement of Theorem 2.3 has been relaxed into the nondegeneracy condition. $\square$

We now turn to the characterization of the dual $G^\vee$ in terms of the spectrum $\Xi(L^1(G))$. To begin we establish a triad of technical lemmas.

Lemma 2.5. *Every multiplicative functional $\varphi : A \longrightarrow \mathbf{C}$ on a (not necessarily unital) Banach algebra A has $\|\varphi\|_{A^*} \leq 1$.*

Proof. The unitization[9] functor U produces a morphism of unital Banach algebras $U\varphi : UA \longrightarrow \mathbf{C}$ which obeys $\|\varphi\|_{A^*} \leq \|U\varphi\|_{(UA)^*}$. Hence without loss of generality we may assume that A is unital.

In this case, suppose for a contradiction that there is some $a \in A$ such that $\|a\|_A < |\varphi(a)|$. Then linearity of φ means that $\varphi(1 - \frac{a}{\varphi(a)}) = 0$ (and $1 \in A$ makes sense by unitality). But on the other hand by hypothesis $\|\frac{a}{\varphi(a)}\|_A < 1$, and so the series $b := \sum_{i=0}^{\infty}(\frac{a}{\varphi(a)})^i$ is absolutely convergent in A. But b is defined by the so-called Neumann[10] series for $(1 - \frac{a}{\varphi(a)})^{-1}$, and in particular witnesses the fact that $1 - \frac{a}{\varphi(a)}$ is invertible in A. Since φ is a multiplicative functional we must then have that $\varphi(1 - \frac{a}{\varphi(a)}) = 0$ is invertible as well, which is the desired contradiction. $\square$

Lemma 2.6. *The topology on $\Xi(L^1(G))$ is locally compact and Hausdorff.*

Proof. This is a standard fact which can be found in any textbook on locally compact abelian groups, for example [22, 13, 6]; the crux is that the topology on $\Xi(L^1(G))$ is that inherited by viewing $\Xi(L^1(G))$ as a subset of the unit ball of $L^\infty(G, \mu)$ equipped with the weak-* topology (note that this is possible by Lemma 2.5). Given this, also note that the same Lemma 2.5 implies by a direct calculation that the set $\Xi(L^1(G)) \cup \{\mathbf{0}\}$—with $\mathbf{0}$ the zero functional on $L^1(G)$—is closed in the weak-* topology. Therefore the Banach–Alaoglu theorem yields that the set $\Xi(L^1(G)) \cup \{\mathbf{0}\}$ is weak-* compact. Hence $\Xi(L^1(G))$ is locally compact, as desired. $\square$

[9]Given A nonunital, one equips $UA := A \oplus \mathbf{C}$ with an algebra structure where $(0, 1) \in A \oplus \mathbf{C}$ is the unit. For a definition and basic properties—such as existence of a norm on UA extending the norm on A—there are many references, for example [5].

[10]Incidentally named for the mathematician Carl Neumann and not John von Neumann.

Lemma 2.7. *The left regular representation commutes with the product in $L^1(G)$, in that for all $f_1, f_2 \in L^1(G)$ and $g \in G$ we have $(g \cdot f_1) * f_2 = f_1 * (g \cdot f_2)$.*

Proof. Fixing $g_1 \in G$, we may compute directly that (using abelianness of G)

$$\begin{aligned}((g \cdot f_1) * f_2)(g_1) &= \int_G f_1(g^{-1}g_2) f_2(g_2^{-1}g_1)\,\mathrm{d}\mu(g_2)\\ &= \int_G f_1(g_2) f_2(g_2^{-1}g^{-1}g_1)\,\mathrm{d}\mu(g_2) = (f_1 * (g \cdot f_2))(g_1),\end{aligned}$$

as desired. □

Theorem 2.8 ([6, 4])**.** *There are mutually inverse continuous maps $\Phi : \Xi(L^1(G)) \leftrightarrows G^\vee : \Psi$. In particular by Lemma 2.6 we have that $G^\vee$ is a locally compact abelian group.*

Proof. Let $\varphi \in \Xi(L^1(G))$ be a nonzero multiplicative functional $L^1(G) \to \mathbf{C}$ (not necessarily a $*$-homomorphism). Since each such φ is assumed nonzero, we may choose some $f_\varphi \in L^1(G)$ such that $\varphi(f_\varphi) \neq 0$. We then define the map $\Phi : \Xi(L^1(G)) \to G^\vee$ by

$$\Phi(\varphi) : g \mapsto \frac{\varphi(g \cdot f_\varphi)}{\varphi(f_\varphi)}.$$

We claim that in fact Φ is well-defined (i.e. $\Phi(\varphi)$ is always an element of $G^\vee$), and that Φ is a bijection (from bijectivity of the inverse Ψ of Φ, with Ψ itself defined without making noncanonical choices, it will follow that the map Φ so constructed is independent of the choice of f_φ for each $\varphi \in \Xi(L^1(G))$ as well).

First, to see that $\Phi(\varphi) \in G^\vee$, we may directly compute for $g_1, g_2 \in G$ that (using Lemma 2.7 and multiplicativity of φ)

$$\begin{aligned}\Phi(\varphi)(g_1 g_2) = \frac{\varphi((g_1 g_2) \cdot f_\varphi)}{\varphi(f_\varphi)} &= \frac{\varphi(g_2 \cdot (g_1 \cdot f_\varphi))}{\varphi(f_\varphi)} \frac{\varphi(f_\varphi)}{\varphi(f_\varphi)}\\ &= \frac{\varphi(g_2 \cdot (g_1 \cdot f_\varphi) * f_\varphi)}{\varphi(f_\varphi)\varphi(f_\varphi)} = \frac{\varphi((g_1 \cdot f_\varphi) * (g_2 \cdot f_\varphi))}{\varphi(f_\varphi)\varphi(f_\varphi)} = \Phi(\varphi)(g_1)\Phi(\varphi)(g_2),\end{aligned}$$

which establishes that $\Phi(\varphi)$ is a group homomorphism.[11] Since $\Phi(\varphi)(g)$ is automatically a continuous function of g, we conclude that Φ indeed maps into $G^\vee$.

It just remains to verify that Φ is a bijection, which we accomplish by constructing an explicit inverse map $\Psi : G^\vee \to \Xi(L^1(G))$. We begin by recalling the canonical isomorphism $\mathbf{C} \cong \mathcal{B}(\mathbf{C})$, and its consequence Lemma 1.8. In particular any $\rho \in G^\vee$, i.e. continuous homomorphism $\rho : G \to \mathbf{T}$, corresponds to an irreducible strongly continuous unitary representation $\widetilde{\rho}$ of G on $\mathbf{C}$. It follows that the machinery of Proposition 1.12 now engages, and yields that integration against $\widetilde{\rho}$ defines a nondegenerate bounded *-representation $\Psi(\rho)$ of $L^1(G)$. In particular, $\Psi(\rho)$ is a nonzero multiplicative functional.

Finally we claim that the maps Φ and Ψ are mutually inverse. Thus first fix $\varphi \in \Xi(L^1(G))$ and $f \in L^1(G)$. By unfolding definitions we then have

$$(\Psi \circ \Phi)(\varphi)(f) = \int_G f(g)\Phi(\varphi)(g)\,\mathrm{d}\mu(g) = \frac{1}{\varphi(f_\varphi)} \int_G f(g)\varphi(g \cdot f_\varphi)\,\mathrm{d}\mu(g). \tag{3}$$

Now, by Lemma 2.5 the map φ is bounded, hence continuous as a linear map $L^1(G) \to \mathbf{C}$. In particular, by the Riesz representation theorem for L^p-spaces, we obtain that there exists $h \in L^\infty(G)$ such that φ is given by integration against h, i.e. for all $f \in L^1(G)$ we have

$$\varphi(f) = \int_G h(g) f(g)\,\mathrm{d}\mu(g).$$

[11] Here we invoke the fact that $\Phi(\varphi)$ actually maps into $\mathbf{T}$ and not $\mathbf{C}$, which is true of any monoid homomorphism from a group into $\mathbf{C}$ (equipped with multiplication).

By Fubini's theorem, this means that we have the identity[12]

$$\begin{aligned}\int_G f(g_1)\varphi(g_1\cdot f_\varphi)\,\mathrm{d}\mu(g_1) &= \int_G\int_G f(g_1)h(g_2)f_\varphi(g_1^{-1}g_2)\,\mathrm{d}\mu(g_2)\,\mathrm{d}\mu(g_1)\\ &= \int_G h(g_2)\int_G f(g_1)f_\varphi(g_1^{-1}g_2)\,\mathrm{d}\mu(g_1)\,\mathrm{d}\mu(g_2)\\ &= \int_G h(g_2)(f * f_\varphi)(g_2)\,\mathrm{d}\mu(g_2)\\ &= \varphi(f * f_\varphi). \end{aligned}\tag{4}$$

Using this in (3), we conclude that $(\Psi\circ\Phi)(\varphi)(f) = \frac{1}{\varphi(f_\varphi)}\varphi(f * f_\varphi) = \varphi(f)$ by multiplicativity of φ. Note that as an additional consequence of (4) we see

$$\int_G (h(g) - \Phi(\varphi)(g))f(g)\,\mathrm{d}\mu(g) = \varphi(f) - \int_G \frac{\varphi(g\cdot f_\varphi)}{\varphi(f_\varphi)}f(g)\,\mathrm{d}\mu(g) = \frac{\varphi(f * f_\varphi)}{\varphi(f_\varphi)} = 0$$

for all compactly supported $f \in C(G)$, from which we conclude that actually $h = \Phi(\varphi)$ in $L^\infty(G)$. That is, φ is given by integration against $\Phi(\varphi)$. But now if $\rho \in G^\vee$, this means that $\Psi(\rho)$ is given by integration against $(\Phi\circ\Psi)(\rho)$. By the definition of Ψ this is just integration against ρ itself, and therefore $(\Phi\circ\Psi)(\rho) = \rho$. This establishes that the maps Φ and Ψ are mutually inverse, and hence completes the proof.[13] □

We now arrive at the main theorem of this section, which—in particular with Theorem 2.8 now in hand—admits a concise proof connecting all of our results thus far.

Theorem 2.9. *For each unitary representation $\rho : G \to \mathcal{B}(\mathcal{H})$ there is a unique $\mathcal{H}$-projection valued measure E on the dual group $G^\vee$ which satisfies*

$$\rho(g) = \int_{G^\vee} \psi(g)\,\mathrm{d}E(\psi). \tag{5}$$

Proof. Let $\rho : G \to \mathcal{B}(\mathcal{H})$ be a unitary representation of G on some Hilbert space $\mathcal{H}$. By Proposition 1.12 ρ corresponds to a nondegenerate *-representation $\Gamma(\rho)$ of $L^1(G)$ (with uniqueness following from Theorem 1.13). By Theorem 2.4 there is in turn a canonically associated $\mathcal{H}$-projection valued measure E on the spectrum $\Xi(L^1(G))$. By Theorem 2.8 there is an isomorphism $\Xi(L^1(G)) \cong G^\vee$, so equivalently we obtain an $\mathcal{H}$-projection valued measure E on $G^\vee$ satisfying (5), as claimed. □

Corollary 2.9.1 (Stone's theorem on one-parameter unitary groups [6, 8, 16])**.** *Let $(\rho_t)_{t\in\mathbf{R}}$ be a strongly continuous one-parameter unitary group on a Hilbert space $\mathcal{H}$. Then there exists a unique self-adjoint linear operator $A : D_A \to \mathcal{H}$ such that*

$$\rho_t = e^{2\pi i t A} \quad \textit{for all } t \in \mathbf{R}, \tag{6}$$

with the exponential defined through the continuous functional calculus for A. Conversely, any self-adjoint linear operator $A : D_A \to \mathcal{H}$ defines a strongly continuous one-parameter unitary group by the formula (6).

Proof. Taking $G = \mathbf{R}$ in Theorem 2.9, we may canonically associate to $(\rho_t)_{t\in\mathbf{R}}$ an $\mathcal{H}$-projection valued measure E on $G^\vee = \mathbf{R}^\vee$. Recalling the canonical isomorphism $\alpha : \mathbf{R} \cong \mathbf{R}^\vee$ defined by $x \mapsto (y \mapsto e^{2\pi i x y})$, the spectral theorem for bounded self-adjoint operators yields that the integral

$$\int_{\mathbf{R}} x\,\mathrm{d}(\alpha^* E)(x)$$

[12] In fact by the same argument we have that every bounded linear functional on $L^1(G)$ commutes with integration in this sense.

[13] Continuity of Φ and Ψ follow directly from a technical argument, using elementary properties of the compact-open topology which we omit, but appears (for example) as Lemma 1.78 of [22]. In order to avoid this detail we could instead have *defined* the topology on $G^\vee$ by transport of structure across Φ.

defines a self-adjoint linear operator A on $\mathcal{H}$. By the functional calculus for such operators for each $t \in \mathbf{R}$ we have another operator, now a bounded linear map on all of $\mathcal{H}$, defined by

$$e^{2\pi i t A} := \int_{\mathbf{R}} e^{2\pi i t x} \, \mathrm{d}(\alpha^* E)(x) = \int_{\mathbf{R}} \alpha(x)(t) \, \mathrm{d}(\alpha^* E)(x) = \int_{\mathbf{R}^\vee} \psi(t) \, \mathrm{d}E(\psi).$$

But by Theorem 2.9 we also have the formula

$$\rho_t = \int_{\mathbf{R}^\vee} \psi(t) \, \mathrm{d}E(\psi),$$

so we necessarily must have $\rho_t = e^{2\pi i t A}$ for all $t \in \mathbf{R}$, as desired.

For the converse we may appeal to Theorem 1.13, but we also have a straightforward direct argument. Starting with a self-adjoint linear operator $A : D_A \to \mathcal{H}$ the fact that the formula (6) defines a strongly continuous one-parameter unitary group is much more elementary; all of the required properties follow directly from elementary properties of the Borel functional calculus. Finally, uniqueness of the self-adjoint operator A also follows directly from a property of the functional calculus for A, since for any $v \in D_A$ the estimate $|e^{2\pi i t x} - 1| \leq 2\pi |tx|$ (with $t, x \in \mathbf{R}$) gives[14] a formula

$$Af = \lim_{h \to 0} \frac{1}{i} \frac{\rho_h v - v}{t}.$$

This completes the proof. □

Stone's theorem (Corollary 2.9.1) is a powerful result, because of how tightly it constrains the initial data of a mere strongly continuous homomorphism $\rho_t : \mathbf{R} \to \mathcal{B}(\mathcal{H})$ into the unitary operators. Indeed, it follows readily from the proof of Corollary 2.9.1 (using the Borel functional calculus) that the map $t \mapsto \rho_t x$ is actually differentiable for all $x \in \mathcal{H}$. There are many applications of Stone's theorem in quantum mechanics, owing to the fact that operators frequently from a one-parameter strongly continuous unitary group (e.g. [8] contains many examples).

There are also purely mathematical consequences, such as the fact that Stone's theorem provides an elegant improvement over the original proof of Bochner's theorem for continuous positive-definite functions on a locally compact abelian group (see for example the proof of Theorem IX.9 of [16]). Finally, we note that more direct proofs avoiding the totality of the locally compact abelian group theory are certainly possible (for example Theorem 10.15 of [8])—though this formulation involving the deduction of Corollary 2.9.1 from the general result Theorem 2.9 naturally unifies Stone's theorem with the (generalized) Stone–von Neumann theorem which we present below.

3 The ECC-relations and the Stone–von Neumann theorem

Stone's theorem admits several generalisations, at least in spirit, for example in the form of the Hille–Yosida theorem (this formulation appears for example as Theorem 3.3.4 of [1]). The Hille–Yosida theorem characterizes the closed linear operators on a Banach space which generate strongly-continuous one-parameter *semigroups*. Another result in a different direction is the celebrated Stone–von Neumann theorem of quantum mechanics, which is our next goal. Glossing over some technical details, the Stone–von Neumann theorem asserts an equivalence between the so-called Heisenberg representation and Schrödinger representation, reconciling the two formulations of quantum mechanics and resolving an open problem of Heisenberg [18].

In this section we give the most basic statement of the Stone–von Neumann theorem using the archetypal example of $\mathbf{R}^n$. We begin with the following physically-motivated definition.

Definition 3.1. A pair of unitary representations ρ and σ of $\mathbf{R}^n$ on a Hilbert space $\mathcal{H}$ are said to satisfy the *exponentiated canonical commutation (ECC-)relations (between position and momentum operators)* if for all $x, p \in \mathbf{R}^n$ we have[15]

$$\rho(x)\sigma(p) = e^{2\pi i x \cdot p} \sigma(p)\rho(x). \tag{7}$$

[14]The details are spelled-out in Proposition 10.14 of [8], but we stress that this is a useful exercise in the application of the Borel functional calculus.

[15]Note that here and throughout we impose a normalization in order to eliminate a constant of $\hbar$, the universal *Planck's constant*.

Example 3.2. As alluded to in the definition, examples of the canonical commutation relations arise from position and momentum operators in quantum mechanics. For instance, setting $n = 1$ we may define linear operators $\widehat{X}$ (*position*) and $\widehat{P}$ (*momentum*) on $L^2(\mathbf{R})$ by setting

$$\widehat{X}(f) = (x \mapsto xf(x)) \quad \text{and} \quad \widehat{P}(f) = \frac{1}{2\pi i}\frac{\partial}{\partial x} f.$$

We let the domain $D_{\widehat{X}}$ of $\widehat{X}$ be set of all $f(x) \in L^2(\mathbf{R})$ such that $xf(x) \in L^2(X)$. Similarly, we let the domain $D_{\widehat{P}}$ of $\widehat{P}$ be the set of all $f(x) \in L^2(\mathbf{R})$ such that $p\mathcal{F}(f)(p) \in L^2(\mathbf{R})$ where $\mathcal{F}(f)(p)$ denotes the Fourier transform of $f(x)$. Being a multiplication operator, $\widehat{X}$ is automatically self-adjoint on this domain, and because the Fourier transform is unitary the operator $\widehat{P}$ is self-adjoint similarly.[16]

With the definitions of $\widehat{X}$ and $\widehat{P}$ in hand, we have in particular for each $x_0 \in \mathbf{R}$ and $f \in C_c^\infty(\mathbf{R})$ that

$$\begin{aligned} 2\pi i \left(\widehat{X}\widehat{P}(f) - \widehat{P}\widehat{X}(f)\right)(x_0) &= x_0 \frac{\partial f}{\partial x}\Big|_{x=x_0} - \frac{\partial}{\partial x}(xf(x))\Big|_{x=x_0} \\ &= x_0 f'(x_0) - f(x) - x_0 f'(x_0) \\ &= f(x_0). \end{aligned} \tag{8}$$

Indeed, we have the same for any $f \in D_{\widehat{X}} \cap D_{\widehat{P}}$ by a simple computation with the Fourier transform. These are the so-called ordinary *canonical commutation relations* $[\widehat{X}, \widehat{P}] = \frac{1}{2\pi i}$ of quantum mechanics [18, 8].

According to one half of Stone's theorem (Corollary 2.9.1), the self-adjoint operators $\widehat{X}$ and $\widehat{P}$ determine strongly continuous one-parameter unitary groups valued in $\mathcal{B}(L^2(\mathbf{R}))$ by the formulas $\rho(x) := e^{2\pi i x\widehat{X}}$ and $\sigma(p) := e^{2\pi i p\widehat{P}}$. In fact, for all $x_0, p_0 \in \mathbf{R}$ we have

$$e^{2\pi i x_0 \widehat{X}} e^{2\pi i p_0 \widehat{P}} = e^{2\pi i x_0 p_0} e^{2\pi i p_0 \widehat{P}} e^{2\pi i x_0 \widehat{X}},$$

i.e. the representations ρ and σ satisfy the exponentiated canonical commutation relations (7). This follows from formally exponentiating (8), but unfortunately cannot be made rigorous in generality.[17] Nonetheless we do actually have explicit formulas (for any $x_0, p_0 \in \mathbf{R}$ and $f \in L^2(\mathbf{R})$)

$$e^{2\pi i x_0 \widehat{X}}(f) = (x \mapsto e^{2\pi i x_0 x} f(x)) \quad \text{and} \quad e^{2\pi i p_0 \widehat{P}}(f) = p_0 \cdot_{\mathrm{lr}} f,$$

where the subscript $\cdot_{\mathrm{lr}}$ is intended to emphasise that p_0 acts on f by the left regular representation of $\mathbf{R}$ on $L^2(\mathbf{R})$. These formulas may be verified by an application of the uniqueness part of Stone's theorem together with the explicit formula for the self-adjoint operator giving rise to a strongly continuous one-parameter unitary group which occurs in the proof. As a consequence for all $x \in \mathbf{R}$ we have

$$\begin{aligned} (e^{2\pi i p_0 \widehat{P}} e^{2\pi i x_0 \widehat{X}} f)(x) &= e^{2\pi i x_0 (x - p_0)} f(x - p_0) \\ &= e^{-2\pi i x_0 p_0} e^{2\pi i x_0 x} f(x - p_0) = (e^{-2\pi i x_0 p_0} e^{2\pi i x_0 \widehat{X}} e^{2\pi i p_0 \widehat{P}} f)(x), \end{aligned}$$

satisfying (7) in this case, as claimed.

Example 3.3. When $n \geq 2$, we can define self-adjoint operators $(\widehat{X}_i)_{1 \leq i \leq n}$ and $(\widehat{P}_i)_{1 \leq i \leq n}$ on $L^2(\mathbf{R}^n)$ coordinate-wise as in the one-dimensional case, i.e. by

$$\widehat{X}_i(f) = (x \mapsto x_i f(x)) \quad \text{and} \quad \widehat{P}_i(f) = \frac{1}{2\pi i}\frac{\partial}{\partial x_i} f,$$

[16] Alternative characterizations of these operators appear in Proposition 9.32 of [8].

[17] One might try to invoke the Baker–Campbell–Hausdorff formula at this point, since the relation $[\widehat{X}, \widehat{P}] = \frac{1}{2\pi i}$ implies that $\widehat{X}$ and $\widehat{P}$ each commute with their commutator. Unfortunately this attempt fails because of unboundedness of $\widehat{X}$ and $\widehat{P}$ on $L^2(\mathbf{R})$. Further, one can actually show that any pair of self-adjoint linear operators A and B satisfying $[A, B] = i$ cannot have either bounded [18, 8] (first stated explicitly in [23] and [21]), so we are out of luck.

with the domains respectively defined as the natural generalizations of the one-dimensional case. We then set $\widehat{X} := \prod_{i=1}^{n} \widehat{X}_i$ and $\widehat{P} := \prod_{i=1}^{n} \widehat{P}_i$, and one easily sees that the operator $\widehat{X}_i$ commutes with the operators $\widehat{X}_j$ and $\widehat{P}_j$ whenever $i \neq j$, and likewise for $\widehat{P}_i$. Exponentiating $\widehat{X}$ and $\widehat{P}$, it can then be checked in exactly the same way that we obtain unitary representations ρ_{m} and σ_{m} now of $\mathbf{R}^n$, which satisfy the exponentiated canonical commutation relations (7).

Being canonically defined, the pair $(\rho_{\mathrm{m}}, \sigma_{\mathrm{m}})$ of representations in Example 3.3 are together known as the *Schrödinger (unitary) representation of* $\mathbf{R}^n$ [22] (the sense in which the pair defines a single representation will become clear in the next section).

We are now equipped with the language required to state the Stone–von Neumann theorem. It is the content of the Stone–von Neumann theorem that, in a precise sense, every pair of unitary representations of $\mathbf{R}^n$ satisfying the exponentiated canonical commutation relations is built from the pair $(\rho_{\mathrm{m}}, \sigma_{\mathrm{m}})$.

Theorem 3.4 (Stone–von Neumann [18])**.** *Let (ρ, σ) be a pair of unitary representations of $\mathbf{R}^n$ on a Hilbert space $\mathcal{H}$ which satisfy the exponentiated canonical commutation relations* (7)*. Then $\mathcal{H}$ is Hilbert space-isomorphic (via a unitary map) to a direct sum of copies of $L^2(\mathbf{R}^n)$, on which the pair (ρ, σ) is taken to a direct sum of copies of the Schrödinger representation $(\rho_{\mathrm{m}}, \sigma_{\mathrm{m}})$.*[18]

The Stone–von Neumann theorem was first explicitly stated by Stone and von Neumann with proofs (of varying degrees of explicitness) given in the period 1930–1931 [18]. The name itself was first coined by Mackey in [11], who recognized that the statement Theorem 3.4 could be generalized to all locally compact abelian groups, a formulation which we will see in the next section. Mackey showed that he could deduce the result by appeal to his *imprimitivity theorem* [12] using the notion of *systems of imprimitivity* of G/H for $H \subset G$ a closed subgroup. A more modern formulation as presented by Rieffel [17] is in terms of representations of C*-algebras, and arises from the observation that systems of imprimitivity correspond to *-representations of certain C*-algebra crossed products. One way to state the Stone–von Neumann theorem in this language is that the crossed product $C_0(G) \rtimes_{\sigma_{\mathrm{m}}} G$ (to be defined in the next section) is C*-Morita equivalent to $\mathbf{C}$, the complex numbers [18]. This follows as a consequence of *Green's imprimitivity theorem* (see e.g. Theorem C.23 of [15], or Theorem 4.21 of [22]), a version of which will appear as Theorem 4.9 below.

4 A modern reformulation: Green's imprimitivity theorem

In this section we present the machinery required to give a modern reformulation and proof of the Stone–von Neumann theorem for all locally compact abelian groups G. We will first need to marshal some powerful structural theorems in the unitary representation theory of locally compact groups, chief among them *Pontryagin duality* (for abelian groups) and *Green's imprimitivity theorem*. We will not have nearly enough time to prove the structural theorems we require, but we will at least explain the elegant way in which they fit together, and the deep connections to what we have seen already in Sections 1 and 2.

Definition 4.1. A pair (ρ, σ) of unitary representations of G and $G^\vee$ respectively, on the same Hilbert space $\mathcal{H}$, is called *Heisenberg*[19] if for all $g \in G$ and $\varphi \in G^\vee$ we have

$$\rho(g)\sigma(\varphi) = \varphi(g)\sigma(\varphi)\rho(g). \tag{9}$$

Note that by the canonical isomorphism $\mathbf{R} \cong \mathbf{R}^\vee$ given by $x \mapsto (y \mapsto e^{2\pi i x y})$, every unitary representation ξ of $\mathbf{R}^n$ is canonically a unitary representation $\widetilde{\xi}$ of $(\mathbf{R}^n)^\vee$. Moreover, one sees

[18]In this way the Schrödinger representation is canonically distinguished; we call a pair (ρ, σ) of unitary representations of $\mathbf{R}^n$ on $\mathcal{H}$ *irreducible* if there are no nontrivial closed subspaces of $\mathcal{H}$ which are simultaneously invariant under all of the operators $\rho(G)$ and $\sigma(G)$. Then the pair $(\rho_{\mathrm{m}}, \sigma_{\mathrm{m}})$ is irreducible, and for any other irreducible pair (ρ, σ) of unitary representations of G on $\mathcal{H}$ satisfying the ECC-relations there is a unitary map $U : \mathcal{H} \to L^2(\mathbf{R}^n)$ which is simultaneously a morphism $\rho \to \rho_{\mathrm{m}}$ and $\sigma \to \sigma_{\mathrm{m}}$ (see Theorem 14.8 of [8]). This naturally generalizes to the situation of the next section.

[19]The name comes from the connection to the Heisenberg group, itself so-named because the canonical generators of its Lie algebra obey the ordinary canonical commutation relations of the previous section. This is Definition 4.25 of [22]; such a pair (ρ, σ) is sometimes also called a *covariant representation* of G, as in [19].

directly that every pair of unitary representations (ρ, ξ) of $\mathbf{R}^n$ on a Hilbert space $\mathcal{H}$ satisfy the exponentiated canonical commutation relations if and only if the pair $(\rho, \widetilde{\xi})$ is Heisenberg (and thus this is a strict generalisation of the case of the previous section). We also have the following example, generalising the Schrödinger representation of $L^2(\mathbf{R}^n)$.

Example 4.2. Recall that for any locally compact abelian group G we have the *left regular representation* of G on $L^2(G)$, given for each $f \in L^2(G)$ and $g \in G$ by

$$\sigma_{\mathrm{m}}(g)(f) := g' \mapsto f(g^{-1}g').$$

Similarly, we also have the canonical representation ρ_{m} of $G^\vee$ on $L^2(G)$ defined for each $f \in L^2(G)$ and $\varphi \in G^\vee$ by

$$\rho_{\mathrm{m}}(\varphi)(f) := g' \mapsto \varphi(g')f(g').$$

(Note that of course $\varphi f \in L^2(G)$ whenever $f \in L^2(G)$ because φ is valued in $\mathbf{T}$.)

Together, it is easy to check explicitly (as we have essentially done above) that the pair $(\sigma_{\mathrm{m}}, \rho_{\mathrm{m}})$ is Heisenberg, and it is called the *Schrödinger representation*[20] *of* G.

Note that if $G = \mathbf{R}^n$ then this notion of Schrödinger representation extends that which was defined previously in Example 3.3. In the same vein, keeping the canonical isomorphism $\mathbf{R} \cong \mathbf{R}^\vee$ in mind, we then have the following direct generalization of Theorem 3.4.

Theorem 4.3 (Stone–von Neumann–Mackey [22, 15]). *Let (ρ, σ) be pair of representations of G on $\mathcal{H}$ and $G^\vee$ on $\mathcal{H}$, respectively. If the pair (ρ, σ) is Heisenberg then ρ and σ jointly decompose as a direct sum of copies of the Schrödinger representation.*

To give a modern proof of Theorem 4.3 we will need to upgrade a few of our previous constructions into their C*-algebraic analogues. Naturally we begin by defining the key C*-algebras which we will need to consider.

Definition 4.4. The *universal enveloping C*-algebra of* $L^1(G)$ is denoted by $C^*(G)$. In other words, $C^*(G)$ is the completion of $L^1(G)$ with respect to the norm $\|f\|_{C^*(G)} := \sup_\rho \|\rho(f)\|$, where ρ ranges over all *-representations of $L^1(G)$ on any Hilbert space. Then $C^*(G)$ is called the *group C*-algebra of* G, and is the natural C*-analogue of the ordinary group algebra.

Definition 4.5. The *C*-algebra of continuous functions on G which vanish at infinity*[21] is denoted by $C_0(G)$.

The relationship between these two constructions—and in particular the utility of passing from $L^1(G)$ to $C^*(G)$—is made clear by the following theorem.

Theorem 4.6 (Fourier–Plancherel–Gelfand [22]). *The Gelfand transform (of Definition 2.2) extends to an C*-isomorphism $C^*(G) \to C_0(G^\vee)$.*

Proof sketch. We have seen that the Gelfand transform naturally gives a map $\widehat{\ } : L^1(G) \to C(\Xi(L^1(G)))$, and by Theorem 2.8 thus a map $\widehat{\ } : L^1(G) \to C(G^\vee)$. In fact the Gelfand transform maps into $C_0(G^\vee)$ by the so-called generalized Riemann–Lebesgue lemma[22] [6]. It can then be shown that the image of the Gelfand transform $L^1(G) \to C_0(G^\vee)$ is dense. Finally, since $C_0(G^\vee)$ is a C*-algebra we obtain a canonical extension to the completion $C^*(G) \to C_0(G^\vee)$ by functoriality of the universal enveloping C*-algebra construction. One then checks that the image of $C^*(G)$ is closed, hence actually all of $C_0(G^\vee)$. $\square$

Since G is abelian, we will also be able to exploit the following famous result.

Theorem 4.7 (Pontryagin duality theorem [6, 10]). *The natural canonical evaluation-at map $G \to G^{\vee\vee}$ is an isomorphism.*

[20] This is Definition 4.25 of [22].

[21] A continuous function $f : X \to \mathbf{C}$ is said to *vanish at infinity* if for all $\varepsilon > 0$ there exists a compact subset $C \subseteq X$ such that $|f(x)| < \varepsilon$ whenever $x \in X \setminus C$.

[22] The name arises from the fact that, using our canonical isomorphism $\mathbf{R} \cong \mathbf{R}^\vee$, the Gelfand transform on $L^1(G)$ is just the Fourier transform—and so in this case this is the ordinary Riemann–Lebesgue lemma.

In order to apply these last two results we require a slight generalisation of our Proposition 1.12, which permits "integrating up" representations of G to nondegenerate *-representations of $C^*(G)$.

Lemma 4.8. *Proposition* 1.12 *extends to the C*-case to give a bijective correspondence between unitary representations of G and nondegenerate *-representations of* $C^*(G)$.

Also note that if H is any closed subgroup of G the quotient space G/H is again a locally compact abelian group (see for instance Section 33 of [10]). In particular there is a canonical Hilbert space $L^2(G/H)$ which recovers $L^2(G)$ when $H = \{1\}$.

The final main ingredient we require is given below, and due its deep connection to the Stone–von Neumann–Mackey theorem is sometimes called the "abstract Stone–von Neumann theorem" [15].

Theorem 4.9 (Green's imprimitivity theorem [22])**.** *For any closed subgroup H of G, there is an isomorphism*[23]

$$C_0(G/H) \rtimes_{\sigma_m} G \cong C^*(H) \otimes \mathcal{K}(L^2(G/H))$$

of C-algebras, where* $C_0(G/H) \rtimes_{\sigma_m} G$ *is the C*-crossed product over the left regular representation of G on*[24] $C_0(G/H)$, *and* $\mathcal{K}(L^2(G/H))$ *is the set of compact operators on* $L^2(G/H)$.

Modulo a minor technical result on the *-representations of algebras of compact operators on Hilbert spaces, we are now ready to prove Theorem 4.3.

Lemma 4.10 ([15])**.** *Let* $\rho : \mathcal{K}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}')$ *be a nondegenerate *-representation on* $\mathcal{H}'$ *of the compact operators on some other Hilbert space* $\mathcal{H}$. *Then there is an indexing set I and a unitary map* $U : \mathcal{H}' \to \oplus_{i\in I}\mathcal{H}$ *which takes* ρ *to a direct sum of copies of the identity representation* $\mathcal{K}(\mathcal{H}) \to \mathcal{K}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$.[25]

Proof of Theorem 4.3. Fix a pair (ρ, σ) of Heisenberg representations on a Hilbert space $\mathcal{H}$. Then σ integrates up to a nondegenerate representation $\widetilde{\sigma}$ of $C^*(G^\vee)$ by Lemma 4.8. Then by an elementary calculation we have that the compatibility condition (9) exactly says that together the pair $(\widetilde{\sigma}, \rho)$ gives a nondegenerate *-representation of the crossed product $C^*(G^\vee) \rtimes_{\sigma_m} G$.

Now, Theorem 4.6 asserts that there is an isomorphism $C^*(G) \cong C_0(G^\vee)$, and hence $C^*(G^\vee) \rtimes_{\sigma_m} G \cong C_0(G) \rtimes_{\sigma_m} G$ by the Pontryagin duality theorem (Theorem 4.7). Taking $H = \{1\}$ in Green's imprimitivity theorem (Theorem 4.9) we obtain a further isomorphism $C_0(G) \rtimes_{\sigma_m} G \cong \mathcal{K}(L^2(G))$. As a consequence of all of these constructions, the Heisenberg pair (ρ, σ) corresponds uniquely to a nondegenerate *-representation of $\mathcal{K}(L^2(G))$. By Lemma 4.10, the resulting representation of $\mathcal{K}(L^2(G))$ must decompose as a direct sum of copies of the identity representation. In particular the Schrödinger representation corresponds to a single copy of the identity representation, and we have only used correspondences which preserve direct sums, so we immediately obtain the desired decomposition of the original pair (ρ, σ). This completes the proof. □

The modern C*-theoretic perspective on the Stone–von Neumann–Mackey theorem has a number of moral successors. A natural generalization arises when one seeks to replace the dual $G^\vee$ in the statement of Theorem 4.3 with another group G' equipped with a nondegenerate pairing $G \times G' \to \mathbf{T}$ (this is a weakening of the case where the pairing is perfect, in which necessarily $G' \cong G^\vee$). The relevant statement was given and proved in full generality by Green [7], which as Rosenberg points out in [18], was also proved or at least exposited in special cases by Pukánszky [14] and Baggett–Kleppner [2]. In addition, there are other nonabelian generalizations of the Stone–von Neumann–Mackey theorem which formulate Green's imprimitivity theorem in the language of dynamical systems—such as of that of [22, 15]—and which require more machinery to state. Of a more physical flavour there is a also a natural supersymmetric generalization [9], where the analogy with the Heisenberg Lie algebra we saw above is replaced with an analogy with the super-Heisenberg Lie algebra. The list goes on, but this essay cannot!

[23] This isomorphism is explicitly constructed in Theorem 4.21 of [22], and in fact does not require that G be abelian (though this is required in order to make sense of all of the previous results of this section). In general if $H \subseteq G$ is closed but not normal the quotient G/H will be locally compact but lack a group structure; however G/H at least admits a so-called regular "quasi-invariant" measure, yielding a well-defined Hilbert space $L^2(G/H)$ in general.

[24] As in the case of $L^1(G)$ and $L^2(G)$, here $g \in G$ acts on $f \in C_0(G/H)$ by $(g \cdot f)(g') := f(g^{-1}g')$. The connection to a Heisenberg pair (σ, ρ) is that, fixing $g \in G$ and $f \in L^1(G)$, with respect to this action the Gelfand transform $\widehat{\ } : C^*(G) \to C_0(G^\vee)$ (as characterized by Theorem 4.6) satisfies $\widehat{g \cdot f}(\psi) = \psi(g)\widehat{f}(\psi)$.

[25] This is Lemma B.34 of [15].

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