Defn. A subbundle of a v.b. $E \xrightarrow{p} B$ is a subspace $F \subseteq E$ s.t. $p|_F^{-1}(x)$ (in topology) is a subspace of E_x, and $F \xrightarrow{p|_F} B$ is again a v.b..

$W \subseteq V$

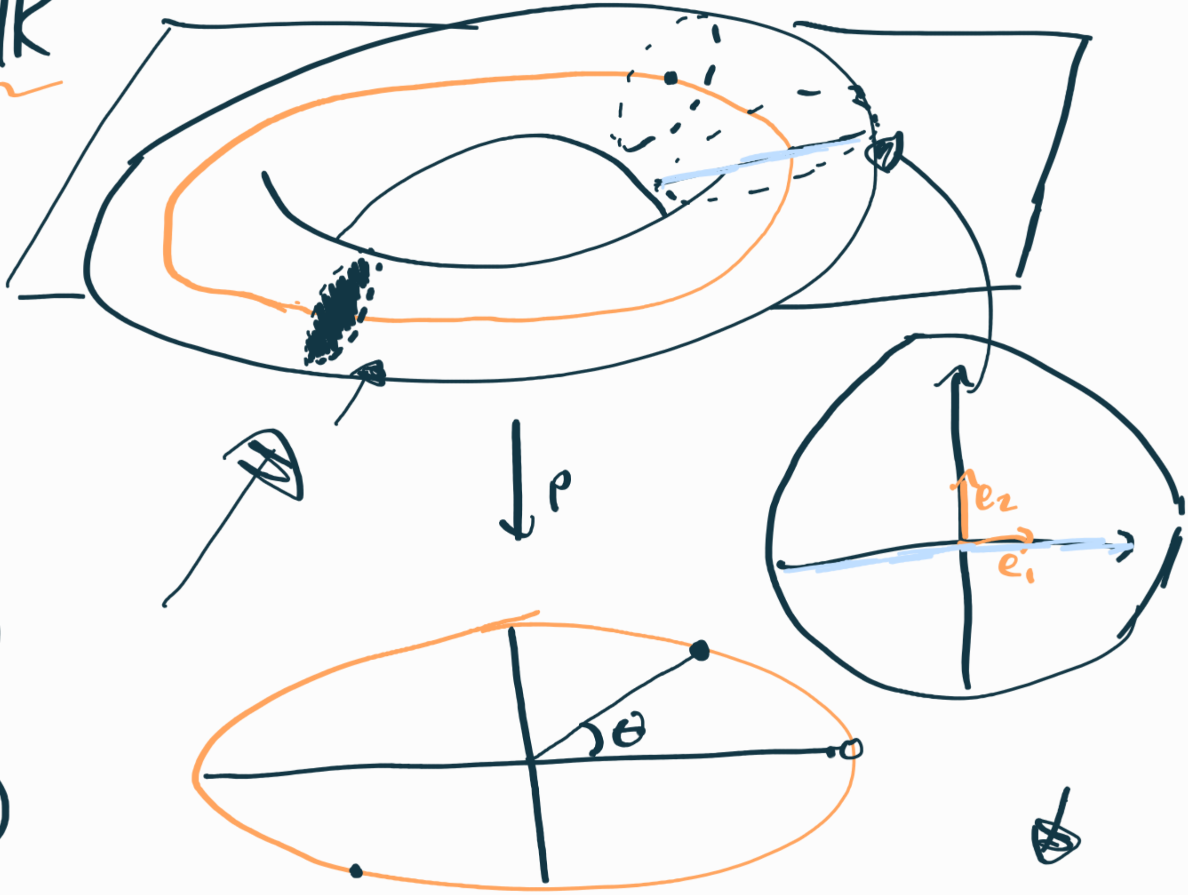
$U, W \subseteq V$

$V = U \oplus W$

(internal) if each $v \in V$ is uniquely $u + w$, with $u \in U$, $w \in W$.

(external) U, W which have nothing to do with V, $U \oplus W$ could still be isomorphic to V.

$S^1 \times \mathbb{R}^2$

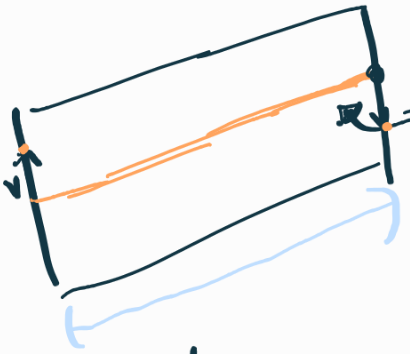


S^1

$[0, 2\pi] \longrightarrow GL(\mathbb{R}^1 \longrightarrow \mathbb{R}^2)$

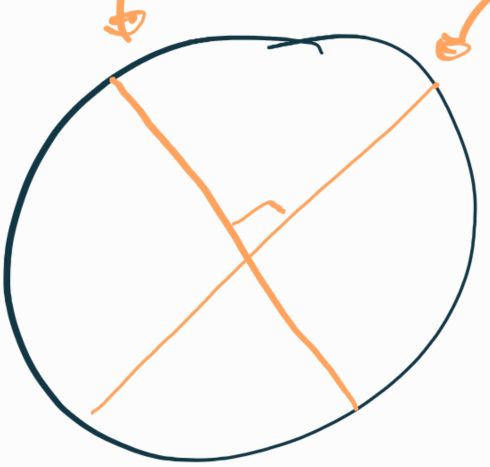
$\phi_1 : \quad \theta \longmapsto \left(t \longmapsto t\left(\cos\frac{\theta}{2} e_1 + \sin\frac{\theta}{2} e_2\right)\right)$

We can build M as

$I \times \mathbb{R}$  $\Big/ (0,v)\sim(1,-v)$ $=$ 

$$\begin{array}{ccc} I\times\mathbb{R} \big/ (0,v)\sim(1,-v) & = & \text{Möbius band} \\ \downarrow & & \downarrow \\ I \big/ 0\sim 1 & = & S^1 \end{array}$$

$$\phi_2 : \theta \longmapsto \left(t \longmapsto t\left(\cos\tfrac{\theta}{2} e_2 + \sin\tfrac{\theta}{2} e_1\right)\right).$$



Punchline $M \oplus M \cong S^1 \times \mathbb{R}^2$.

Observation For v.s.

① $U \subseteq V$ is always complemented, so $U \oplus U' \cong V$.

② The form $U' = \text{span}\{v_1, \ldots, v_n\}$,

$\mathcal{B}_U = \{u_1, \ldots, u_m\}$ $\mathcal{B}_V = \{u_1, \ldots, u_m, v_1, \ldots, v_n\}$.

We say a subobject $U \subseteq X$ is complemented if $\exists\, U' \subseteq V$ s.t. $V \cong U \oplus U'$.

② The complement U' is not canonical.

③ Nonetheless, there is a canonical space which U' is isomorphic to:

$$V/U$$

$\hookrightarrow$ $V \cong U \oplus V/U$.

noncanonical

④ But, the complement U' is canonically determined when V has an inner product.

$\hookrightarrow$ $U' := U^{\perp}$.

What about for v.b.?

$$\begin{tikzcd} F \arrow[dr] & \subseteq & E \arrow[dl] \\ & B & \end{tikzcd} \qquad F_x \subseteq E_x$$

<u>Defn.</u> An <u>inner product</u> on $\begin{array}{c} E \\ \downarrow \\ B \end{array}$ is a choice of inner product on each fiber E_x such that there exist local trivializations

$$\phi : E_U \xrightarrow{\sim} U \times \mathbb{R}^n \qquad (*)$$

$$\cup \qquad E_x \longrightarrow \{x\} \times \mathbb{R}^n$$

which take the inner product on $E_x \subseteq E_U$ to the standard inner product on $\mathbb{R}^n$.

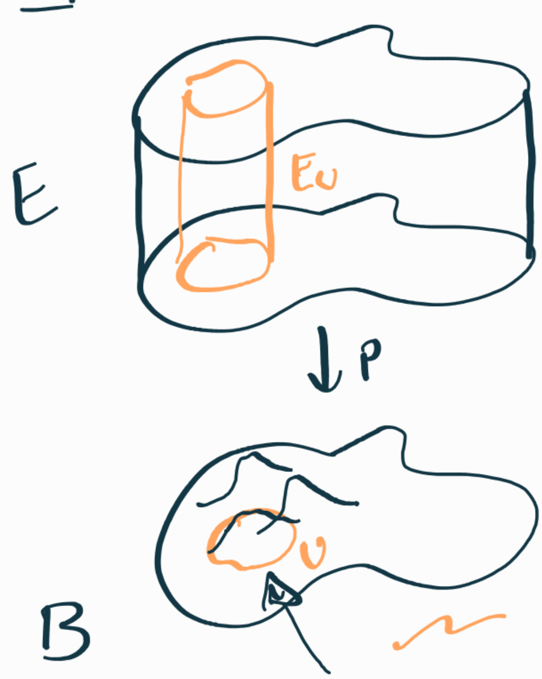
$\hookrightarrow \otimes$ Or, a <u>v.b. map</u>

$$\begin{tikzcd} E \otimes E \arrow[rr] \arrow[dr] & & B \times \mathbb{R} \arrow[dl] \\ & B & \end{tikzcd}$$

s.t. the induced maps $E_x \otimes E_x \longrightarrow \{x\} \times \mathbb{R}$ is <u>always</u> an inner product.

Prop. When B is Hausdorff & paracompact, all v.b.s over B have inner products.

Pf.



$\phi: E_U \xrightarrow{\sim} U \times \mathbb{R}^n$

has an i.p.

because this does.

Pick an l.o.c. of B, call it $\{U_\alpha\}$.

Then there exist $g_\alpha : B \longrightarrow [0,1]$ s.t.

① $g_\alpha|_{U_\alpha^c} = 0$

② $\forall x \in B, \sum_\alpha g_\alpha(x) = 1.$

(and, this sum is always finite.)

(actually, each $x \in B$ has an open $U \subseteq B$ containing it, on which the same merely finite collection of g_α's are nonzero.)

③

Pick $\phi_\alpha : E_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{R}^n$, hence obtain i.p. on the E_{U_α}s. Call this $\langle -,- \rangle_\alpha : E_{U_\alpha} \otimes E_{U_\alpha} \to U_\alpha \times \mathbb{R}$.

Now simply define.

on E_{U_α}

$$\langle \underset{\in E_y}{\overset{v}{-}}, \underset{\in E_y}{\overset{v'}{-}} \rangle := \sum_\alpha g_\alpha(x) \langle -,- \rangle_\alpha .$$

$$\begin{tikzcd} E \otimes E \arrow[rr] \arrow[dr] & & B \times \mathbb{R} \arrow[dl] \\ & B & \end{tikzcd}$$

Take $y \notin U_\alpha$.

Fact. If $\langle -,- \rangle_1$ & $\langle -,- \rangle_2$ are inner products on V, then positive linear combos

$$t\langle -,- \rangle_1 + s\langle -,- \rangle_2 ,$$

$t, s > 0$, are again inner products.

∎

Corollary. Every $F \subseteq E$ is complemented, when B is Hausdorff, paracompact.

$$\begin{matrix} F & \subseteq & E \\ & \searrow \swarrow & \\ & B & \end{matrix}$$

Pf. Pick an inner product on E. Then define

$$F^{\perp} := \{ e \in E : \langle e, f \rangle = 0 \ \forall f \in F_{p(e)} \}.$$

$$\begin{array}{c} F^{\perp} \\ \big\downarrow {\scriptstyle p|_{F^{\perp}}} \\ B \end{array}$$

It's clear that $p|_{F^{\perp}}^{-1}(x) = (F_x)^{\perp}$.

$\boxed{A}$ View the inner prod. on E as a map of v.b.s not $E \otimes E \longrightarrow B \times \mathbb{R}$, but instead $E \xrightarrow[\Phi]{} E^*$.

↳ Just like for ordinary v.s.

$U \subseteq V$

$$\langle -, - \rangle : V \otimes V \longrightarrow \mathbb{R}$$

$$\begin{aligned} \overset{\shortparallel}{\Phi} : \quad V &\longrightarrow V^* \longrightarrow U^* \\ v &\longmapsto (v' \longmapsto \langle v, v' \rangle). \end{aligned}$$

$$\begin{aligned} \ker \Phi &= \{ v \in V : \Phi(v)|_U = 0 \}. \\ &= \{ v \in V : \forall v' \in U, \ \underline{\langle v, v' \rangle} = 0 \}. \\ &= U^{\perp} \end{aligned}$$

Returning back to v.b.-land, we get

$$E \xrightarrow{\Phi} E^* \xrightarrow{\pi_F^*} F^*.$$

Define $F^\perp := \ker(\pi_F^* \circ \Phi)$.

It suffices to show.

$$\begin{tikzcd} \ker f \arrow[r, hook] \arrow[drr] & E \arrow[r, "f"] \arrow[dr] & F \arrow[d] \\ & & B \end{tikzcd}$$

$(\ker f)_x := \ker(f : E_x \to F_x)$.

$\boxed{B}$ Given $x \in B$, then we can find.

$\phi_x : E_{U_x} \xrightarrow{\sim} \underline{U_x \times \mathbb{R}^n}$, $x \in U_x$

$\cup$

$F^\perp_{U_x} \xrightarrow{\sim}$

$$\phi_x^{-1} : U_x \times \mathbb{R}^n \xrightarrow{\sim} E_{U_x}.$$

$$(y, v) \longmapsto \_\_\_\_ \in E_y.$$

$\underline{e_1, e_2, \dots, e_n} \mapsto \phi_x^{-1}(y, e_1), \dots, \phi_x^{-1}(y, e_n).$

$$U_x \longrightarrow E_{U_x}$$

$$\sigma_i : y \longmapsto \phi_x^{-1}(y, e_i)$$

$\{\sigma_1, \dots, \sigma_n\}$ everywhere linearly indep. section.

Recall: $F_{U_x} \subseteq E_{U_x}$.

$V \subseteq Y$

$\{u_1, \dots, u_n\}$ $\quad \{u_1, \dots, u_n, v_1, \dots, v_k\}$.

$U^\perp$ basis?

G-S!

$$\begin{tikzcd} F_{U_x \cap V_x} \arrow[d] & F_{U_x} \subseteq E_{U_x} \arrow[d] \arrow[d, shift right=4ex] & E_{U_x \cap V_x} \arrow[d] \\ U_x \cap V_x & U_x \subseteq B. & U_x \cap V_x. \end{tikzcd}$$

We may assume $F_{U_x} \subseteq E_{U_x}$ is a triv. subbundle.

$$\begin{tikzcd} \phi : E_{U_x} \arrow[r] & U_x \times \mathbb{R}^n \arrow[r] & U_x \times \mathbb{R}^n \\ & & \\ F_{U_x} \arrow[u, phantom, "\cup"] \arrow[uur, dotted, "?"] \arrow[rr] & & U_x \times \mathbb{R}^m \arrow[uu, hook] \end{tikzcd}$$

$$\{\underbrace{\sigma_1, \dots, \sigma_m}_{\hat{F}_{U_x}}, \underbrace{\tau_1, \dots, \tau_{n-m}}_{\hat{E}_{U_x}}\}.$$

Do G-S... e.g. $\sigma_1'(x) := \frac{\sigma_1(x)}{\sqrt{\langle \sigma_1(x), \sigma_1(x) \rangle}}$

$\vdots$

But, now $\{\tau_1, \dots, \tau_{n-m}\}$ gives $n-m$ linearly indep. sections of $\begin{array}{c} F^{\perp}_{U_x} \\ \downarrow \\ U_x \end{array}$.

$(\Longrightarrow)$ i.e. a local triv. ☒

Prop. If B is compact Hausdorff, then every v.b. over B is a summand of a trivial bundle.

$$\left(\begin{array}{c} E \\ \downarrow \\ B \end{array} \oplus \begin{array}{c} E' \\ \downarrow \\ B \end{array} \cong \begin{array}{c} B\times\mathbb{R}^n \\ \downarrow \\ B \end{array} \right)$$

Serre–Swan theorem.

Pf. It suffices to produce a trivial bundle of which $\begin{array}{c} E \\ \downarrow \\ B \end{array}$ is a subbundle.

$\vdots$

For next time. $\square$

Observations.

- We can form the set

$$\mathrm{Vect}^k(B) := \{\text{iso. classes of v.b. over } B\}.$$

- When $k=1$, $\mathrm{Vect}^1(B)$ is an abelian group. ← line bundles.
    - $\otimes$ is the grp. op.
    - e is the (trivial) bundle $B \times \mathbb{R} \to B$
    - $[L]^{-1} := [L]$.

Ex. • Show $(B \times \mathbb{R} \to B) \otimes (E \to B)$ is canonically iso. to E. (E of any rank k)

$(\mathbb{R} \otimes V \cong V)$.

• Show $L \otimes L \cong B \times \mathbb{R}$.