<u>Recall</u>. We wanted to compute $K(S^2)$.

↳ More generally

<u>Fundamental Product Theorem</u>

There is a ring iso:

$$K(B) \otimes \mathbb{Z}[H] \big/ (H-1)^2 \longrightarrow K(B \times S^2) \qquad (*)$$

<u>$\otimes$-prod. rings</u>

Given rings R & S, we get a new ring $R \otimes S$ by forming the free abelian group on pairs (r, s), $r \in R$, $s \in S$

$r \otimes s$

and then declaring that... "the usual addition and multiplication laws are obeyed".

$$\bullet \quad \underline{r\otimes(s+s')} = \underline{r\otimes s} + \underline{r\otimes s'}$$

$$\cap$$

$$\mathbb{Z}[\{r\otimes s : r\in R, s\in S\}].$$

We get a multiplication by

$$(r\otimes s)\cdot(r'\otimes s') = rr'\otimes ss'.$$

The map $(*)$ is

$$\begin{array}{ccccc} K(B) & = & K(B) & & \\ \otimes & \longrightarrow & \otimes & \xrightarrow[\otimes]{*} & K(B\times S^2) \\ \mathbb{Z}[H]/(H-1)^2 & \longrightarrow & K(S^2) & & \end{array}$$

$$H \longmapsto$$

Defn The external (tensor) product of v.b. $E \xrightarrow{p} B$ and $F \xrightarrow{q} C$ is a new v.b. over $B \times C$:

$$E * F := pr_1^* E \otimes pr_2^* F$$

$$\begin{tikzcd} & B \times C \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ B & & C \end{tikzcd}$$

$$(E * F)_{(b,c)} = E_b \otimes F_c$$

Over $(b,c) \in B \times C$, what does $(E * F)_{(b,c)}$ look like?

Above $b \in B$ sits E_b, so what does the fiber $(pr_1^* E)_{(b,c)}$ look like?

$$= E_b$$

$$pr_1^* E \subseteq (B \times C) \times E$$

$$\{ ((b,c), e) : pr_1(b,c) = p(e) \}$$

$(b,c) \in B \times C$, $e \in E$, $pr_1(b,c) = b$

If $B = C$, then we have a $\Delta : B \to B \times B$,
$b \mapsto (b, b)$

and $E \times F$ is a v.b. over $B \times B$. So, we can form

$$\Delta^*(E \times F)_b = E_b \otimes F_b.$$

<u>Ex.</u> $\Delta^*(E \times F) \cong E \otimes F$.

<u>Rk.</u> Sometimes Hatcher calls this map μ.

<u>Corollary</u> of FPT:

$$K(S^2) = \mathbb{Z}[H]/(H-1)^2$$

<u>Pf.</u> Take $K(*)$. β $R \otimes \mathbb{Z} \cong R$. $\blacksquare$

$(V \otimes \mathbb{R} \cong V)$

<u>Defn</u> A sequence of maps

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \longrightarrow \cdots \xrightarrow{f_{n-1}} A_n$$

is <u>exact</u> if $\ker f_i = \operatorname{im} f_{i-1}$.

Ex ① $0 \to U \hookrightarrow V \to V/U \to 0$ is an exact sequence.

② $0 \to V \to V \oplus W \to W \to 0$. — exact sequence.

Defn. A short exact sequence (SES) is an exact seq. exactly of the form

$$0 \to A \to B \to C \to 0.$$

We say that B is an extension of C by A.

Lemma If $0 \to A \to B \to C \to 0$ is an exact sequence then $C \cong B/\operatorname{im} A$.

Ex Check this (is the FIT for groups/v.s. etc.)

For you to do:

If $0 \to A \xhookrightarrow{f} B \overset{g}{\twoheadrightarrow} C \to 0$ is a SES, then

① What can we say about the maps f & g?

$\ker f = 0$ (f is injective).

$\rightarrow$ $\operatorname{im} f = \ker g$

$\operatorname{im} g = C$. (g is surjective)

<u>Defn.</u> A SES $0 \to A \to B \to C \to 0$ is <u>split</u> if $B \cong A \oplus C$.

<u>Prop.</u> (Splitting lemma)

A SES $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is

split $\iff$ f has a section $\iff$ g has a section.

(A section σ of $A \xrightarrow{f} B$ is a map $B \to A$ s.t. $f \circ \sigma = id_B$.)

Pf. Check it! $\blacksquare$

Sources of examples of non-split SES/nontrivial extensions include semidirect products of groups ... or more interesting domain specific examples e.g. in AT.

Consider a closed subspace $A \subset X$, X a cpt. Hausdorff top. space.

$$A \overset{\iota}{\hookrightarrow} X \xrightarrow{\pi} X/A$$

$$\tilde{K}(A) \xleftarrow{\tilde{K}(\iota) = \iota^*} \tilde{K}(X) \xleftarrow{\tilde{K}(\pi) = \pi^*} \tilde{K}(X/A)$$

Prop. This sequence is exact at $\tilde{K}(X)$. ($\ker \iota^* = \operatorname{im} \pi^*$).

Pf. ($\operatorname{im} \pi^* \subseteq \ker \iota^*$): Suppose that a is a class in $\operatorname{im} \pi^*$, i.e. $a = \pi^* b$ for some $b \in \tilde{K}(X/A)$. (Need that $\iota^* \pi^* b = 0$ in $K(A)$.)

$$
\begin{tikzcd}
A \arrow[r, hook] \arrow[d] & X \arrow[d] \\
A/A \arrow[r, hook] & X/A
\end{tikzcd}
$$

$$\rightsquigarrow$$

$$
\begin{tikzcd}
 & \tilde{K}(A) & \tilde{K}(X) \arrow[l, "\iota^*"'] \\
0 \cong & \tilde{K}(A/A) \arrow[u] & \tilde{K}(X/A) \arrow[l] \arrow[u, "\pi^*"']
\end{tikzcd}
$$

$\iota^* \circ \pi^* = 0$.

($\operatorname{im} \pi^* \subseteq \ker \iota^*$): Let $E \to X$ be a v.b. which is trivial on A after forming a direct sum with a trivial bundle.

Pick $h : (E \oplus \varepsilon^n)_A \xrightarrow{\sim} A \times \mathbb{C}^k$, a local trivialization.

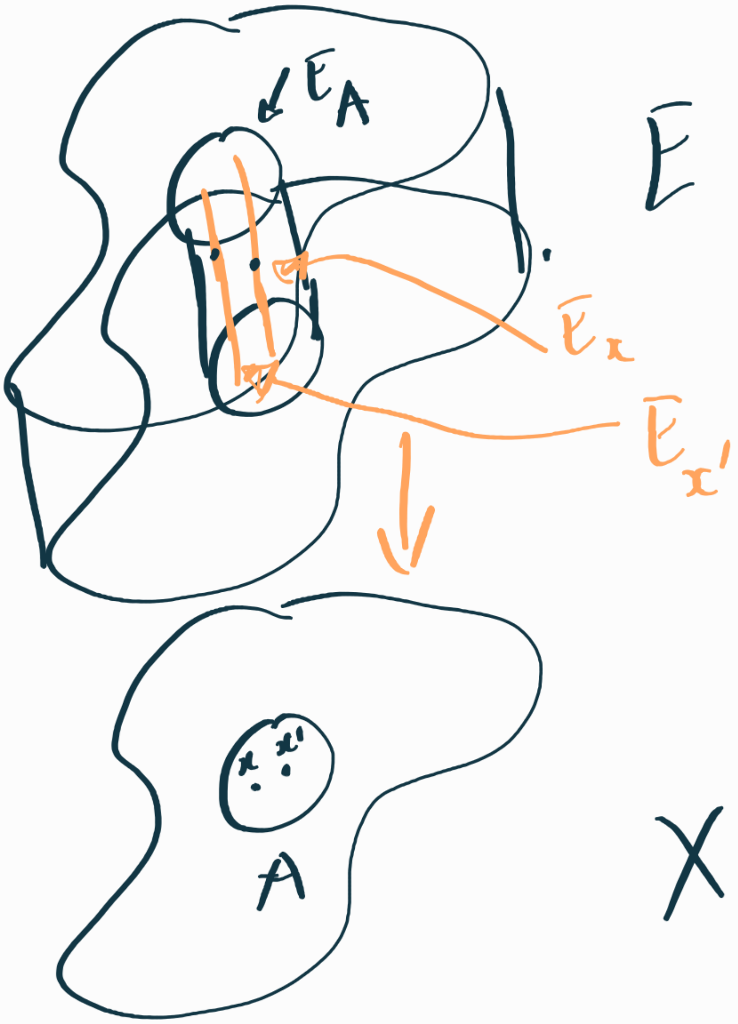
$$\begin{tikzcd}
(E\oplus\varepsilon^n)_A \arrow[rr, "\sim"] \arrow[dr] & & A \times \mathbb{C}^k \arrow[dl] \\
& A &
\end{tikzcd}$$

Replace E with $E \oplus \varepsilon^n$.

Form the space

$$E/h = E \Big/ \underset{\substack{\uparrow \\ E_x}}{e} \sim \underset{\substack{\uparrow \\ E_{x'}}}{e'}$$

whenever $x, x' \in A$ and $pr_2(h(e)) = pr_2(h(e'))$.



Ex Show that to have local trivializations, it suffices to just check that E is trivial on a nbhd of A.

The data of h is equivalent to a family of k-(everywhere) linearly indept. sections of A. Call them

$$\{\sigma_1, \ldots, \sigma_k\}, \text{ maps.}$$

$$\sigma_i : A \longrightarrow E.$$

We can continuously extend the σ_i's to give ~~sections~~ of E over a nbhd U of A in X.

* (But we lose the fact that they are all everywhere linearly indep.)

Pick an open cover of $A \subseteq X$, $\{U_j\}$ over which X is trivial, so we have.

$$h_j : E_{U_j \cap A} \xrightarrow{\sim} (U_j \cap A) \times \mathbb{C}^k$$

We can extend $\sigma_i|_{U_j \cap A} : U_j \cap A \to E$ to a section (actually allowed) $\tilde{\sigma}_{i,j} : U_j \longrightarrow E$.

(But remember *.)

Pick $\{\psi_j\}$, partitions of unity subordinate to the open cover $\{U_j\}$.

Standard trick: The sum

$$\tilde{\sigma}_i := \sum_j \psi_j \tilde{\sigma}_{i,j}$$

actually makes sense on all of $\bigcup U_j$.

open nbhd of A in X.

By construction $\tilde{\sigma}_i \big|_A = \sigma_i$.

The determinant of $\begin{bmatrix} \tilde{\sigma}_1 & \cdots & \tilde{\sigma}_k \end{bmatrix}$ is nonzero on A, and hence also in an open nbhd.

So by suitably restricting $\bigcup_i U_{v_i}$, we obtain the desired nbhd. $\blacksquare$

Upshot. We have

$$\begin{tikzcd}
 & & X \cup CA \arrow[r] \arrow[d] & (X \cup CA) \cup CX \arrow[r] \arrow[d] & \cdots \\
A \arrow[r, hook] & X \arrow[r] \arrow[ur] & X/A \arrow[r, dashed, "\tilde{K}"] & SA &
\end{tikzcd}$$

$$\tilde{K}(A) \leftarrow \tilde{K}(X) \leftarrow \tilde{K}(X/A) \leftarrow \tilde{K}(SA)$$

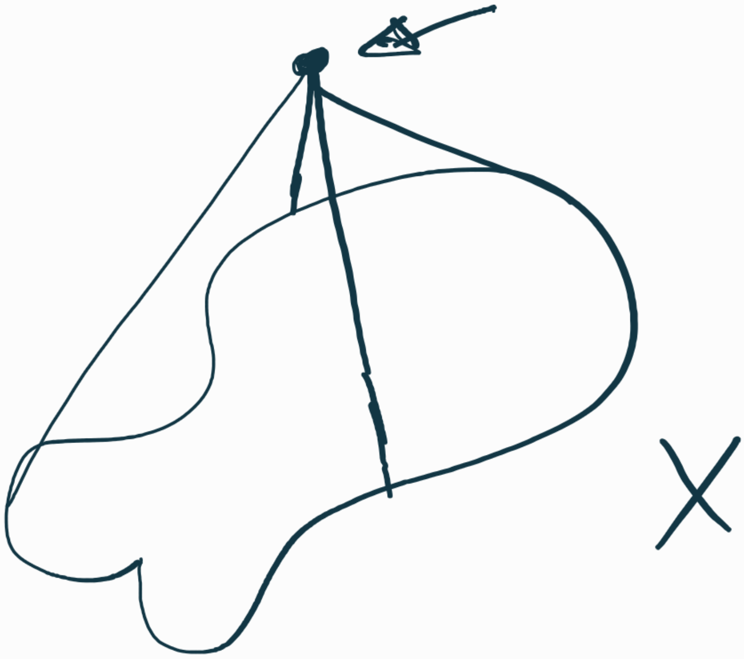
$$\leftarrow \tilde{K}(SX) \leftarrow \tilde{K}(S(X/A))$$

$$\leftarrow \tilde{K}(S^2A) \leftarrow \cdots$$

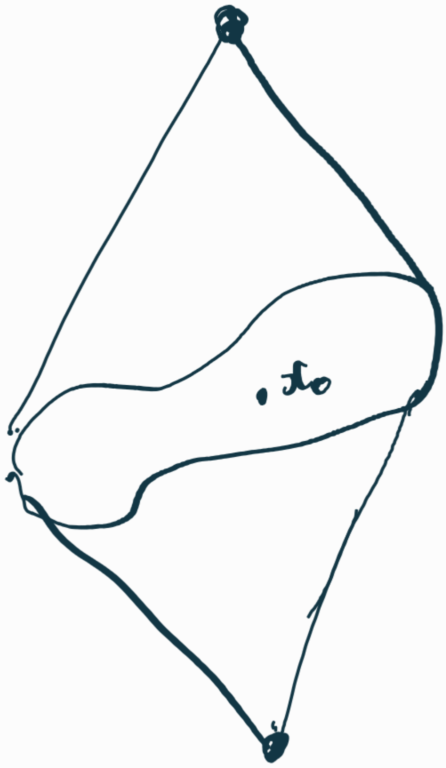
$$\big((X \cup CA) \cup CX\big) \cup C(X \cup CA)$$

Ex: Convince yourself that when we collapse $X \cup CA$ we recover SX.

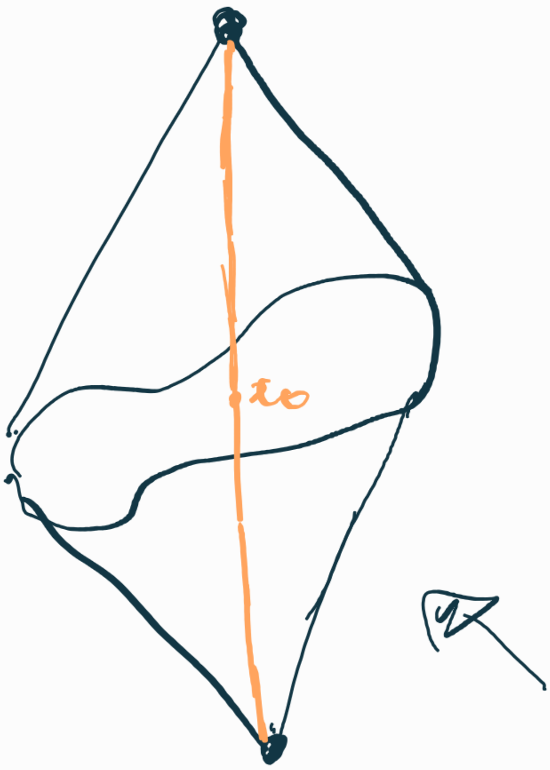
<u>Defn</u>. The <u>cone</u> CX on X is just $X \times I \big/ (x,1) \sim (x',1)\ \forall x, x' \in X$,



<u>Defn</u>. The suspension SX of X is $X \times [-1,1] \big/ \begin{matrix} (x,1) \sim (x',1)\ \forall x, x' \in X \\ (x,-1) \sim (x',-1)\ \forall x, x' \in X \end{matrix}$



Defn. Let (X, x_0) be a pointed topo. space. Then the **reduced suspension** ΣX is



$$SX \Big/ (x_0, t) \sim (x_0, t') \quad \forall t'.$$

<u>Lemma</u>. Let $A \subseteq X$ be closed and contractible. The quotient map $\pi : X \to X/A$ induces an isomorphism of the sets

$$\mathrm{Vect}(X/A) \xrightarrow{\sim} \mathrm{Vect}(X).$$