WHO PUT ANALYSIS IN MY ALGEBRA?

In this talk, I'd like to tell you a few things about doing analysis in positive characteristic. I'd like to begin with some motivation via Hodge theory, which was, and still is, a major driving force for a lot of modern math.

1. Hodge theory via analysis

Let X be a complex manifold. The almost complex structure on X allows us to write $T_X = T_X^{1,0} \oplus T_X^{0,1}$. Let $\Omega_X^{1,0}$ denote the linear dual of $T_X^{1,0}$, and similarly for $\Omega_X^{0,1}$. Taking the linear dual of this splitting of T_X, and then taking the kth exterior power gives a splitting

$$\Omega_X^k \cong \bigoplus_{p+q=k} \Omega_X^{p,q},$$

where $\Omega_X^{p,q} = \wedge^p \Omega_X^{1,0} \oplus \wedge^q \Omega_X^{0,1}$. In words: every C^∞-differential k-form can be written as a uniquely as a sum of differential forms that look like

$$f dz_1 \wedge \cdots \wedge dz_p \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_q \in \Omega_X^{p,q},$$

with $p+q=k$. The $(n,0)$-forms are called holomorphic n-forms on X.

Our goal will be to understand the relationship between this decomposition of the k-forms on X and the de Rham cohomology of X, with coefficients in $\mathbf{C}$. Recall that $\mathrm{H}^k_{\mathrm{dR}}(X;\mathbf{C})$ is the cohomology of the complex

$$\mathbf{C} \otimes_{\mathbf{R}} (\Omega_X^0 \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \to \cdots)$$

The coordinates z and $\overline{z}$ allow us to form new differentials that act on these (p,q)-forms:

$$\partial : \Omega_X^{p,q} \to \Omega_X^{p+1,q}, \ \overline{\partial} : \Omega_X^{p,q} \to \Omega_X^{p,q+1}.$$

For each p, we can therefore look at the resulting complex

$$\Omega_X^{p,0} \xrightarrow{\overline{\partial}} \Omega_X^{p,1} \xrightarrow{\overline{\partial}} \Omega_X^{p,2} \to \cdots$$

We can then take the qth cohomology $\mathrm{H}^q(X;\Omega_X^{p,\bullet})$ of this complex; this is called the *Dolbeault* cohomology of X, and is denoted $\mathrm{H}^{p,q}_{\mathrm{Dol}}(X)$. The main theorem of Hodge theory says:

Theorem 1.1. *Suppose X is a compact Kähler manifold. Then*

$$\mathrm{H}^k_{\mathrm{dR}}(X;\mathbf{C}) \cong \bigoplus_{p+q=k} \mathrm{H}^{p,q}_{\mathrm{Dol}}(X).$$

Let us describe the general recipe for proving this. For notational convenience, we will just write Ω_X^k to denote its sheaf of sections.

Recipe 1.2. (a) Suppose (X,g) is a compact closed Riemannian n-manifold with a chosen orientation. This allows us to define the Hodge star operator $* : \Omega_X^k \to \Omega_X^{n-k}$, and hence an inner product on Ω_X^k: given $\alpha, \beta \in \Omega_X^k$, define

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge \star\beta.$$

One can check that the adjoint (under this inner product) to the differential $d : \Omega_X^k \to \Omega_X^{k+1}$ is given by the map $*d* : \Omega_X^k \to \Omega_X^{k-1}$ (up to some sign); we will denote it by d^*.

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(b) Define the Laplacian $\Delta : \Omega_X^k \to \Omega_X^k$ by

$$\Delta = dd^* + d^* d.$$

Suppose $k = 0$; then d^* is zero (since it lands in the zero group), and so if f is a smooth function on X, then we can do the following calculation in local coordinates:

$$\Delta f = d^* df = d^* \sum_{i=1}^{n} (\partial_i f) dx_i = -\sum_{i=1}^{n} \partial_i^2 f,$$

justifying the name.

(c) Say that a form $\alpha \in \Omega_X^k$ is harmonic if $\Delta\alpha = 0$, and let $\mathrm{H}_\Delta^k(X)$ denote the space of harmonic k-forms. Then, one shows that every cohomology class in $\mathrm{H}_{\mathrm{dR}}^k(X)$ admits a canonical representative in $\mathrm{H}_\Delta^k(X)$[1]. In fact, since $\ker(\Delta) = \ker(d) \cap \ker(d^*)$, this is a consequence of an orthogonal decomposition

$$\Omega_X^k \cong \mathrm{H}_\Delta^k(X) \oplus \mathrm{im}(d) \oplus \mathrm{im}(d^*).$$

This lies at the heart of Hodge theory, and can be proved by some analysis using elliptic operators.

(d) Note that we can identify

$$\ker(d) = \mathrm{H}_\Delta^k(X) \oplus \mathrm{im}(d),\ \ker(d^*) = \mathrm{H}_\Delta^k(X) \oplus \mathrm{im}(d^*).$$

(Included in this is a miraculous fact: the space $\mathrm{H}_\Delta^k(X)$ is finite-dimensional.) In particular, $\mathrm{H}_\Delta^k(X)$ is the orthogonal complement of $\mathrm{im}(d)$ in $\ker(d)$, which means that the composite

$$\mathrm{H}_\Delta^k(X) \subseteq \ker(d) \twoheadrightarrow \ker(d)/\mathrm{im}(d) = \mathrm{H}_{\mathrm{dR}}^k(X)$$

is an isomorphism.

(e) We haven't yet used the complex structure. Suppose (X, h) is now a compact closed complex manifold of complex dimension n, with a Hermitian metric h. We will now implicitly complexify our sheaves of k-forms. Just as with d^* above, we can construct formal adjoints

$$\partial^* : \Omega_X^{p,q} \to \Omega_X^{p-1,q},\ \overline{\partial}^* : \Omega_X^{p,q} \to \Omega_X^{p,q-1}.$$

Moreover, we can again define Laplacians

$$\Delta_\partial = \partial\partial^* + \partial^*\partial : \Omega_X^{p,q} \to \Omega_X^{p,q},$$

and similarly for $\Delta_{\overline{\partial}}$. We can then make a few definitions:

$$\mathrm{H}_{\overline{\partial}}^k(X) = \ker(\Delta_{\overline{\partial}} : \Omega_X^k \to \Omega_X^k),\ \mathrm{H}_{\overline{\partial}}^{p,q}(X) = \ker(\Delta_{\overline{\partial}} : \Omega_X^{p,q} \to \Omega_X^{p,q}),$$

and similarly for $\mathrm{H}_\partial^k(X)$ and $\mathrm{H}_\partial^{p,q}(X)$.

(f) Just as in the "smooth" case above, one can show that there are orthogonal decompositions

$$\Omega_X^{p,q} \cong \mathrm{H}_{\overline{\partial}}^{p,q}(X) \oplus \mathrm{im}(\overline{\partial}^*) \oplus \mathrm{im}(\overline{\partial}),$$

and similarly

$$\Omega_X^{p,q} \cong \mathrm{H}_\partial^{p,q}(X) \oplus \mathrm{im}(\partial^*) \oplus \mathrm{im}(\partial).$$

[1] To see this, consider the space $L^2(\Omega_X^k)$ obtained by completing $C^\infty(X; \Omega_X^k)$ with respect to the above inner product. This is a Hilbert space, and so for any $\alpha \in C^\infty(X; \Omega_X^k)$, the closure of the subspace of $L^2(\Omega_X^k)$ defined by the de Rham cohomology class of α is a closed subspace. It therefore has a unique element of minimal norm. This form can be checked to be killed by d and d^*, and hence is harmonic.

(g) To relate these different decompositions, we now use the Kähler assumption: this implies the famous Kähler identity

$$\Delta = 2\Delta_\partial = 2\Delta_{\overline{\partial}}.$$

Since $\mathrm{H}^k_\Delta(X)$ was defined as $\ker(\Delta)$, we see that

$$\mathrm{H}^{p,q}_\partial(X) \cong \mathrm{H}^{p,q}_{\overline{\partial}}(X),\ \mathrm{H}^k_\Delta(X) \cong \bigoplus_{p+q=k} \mathrm{H}^{p,q}_\partial(X).$$

(h) Combining with our isomorphism $\mathrm{H}^k_{\mathrm{dR}}(X;\mathbf{C}) \cong \mathrm{H}^k_\Delta(X)$, we see

$$\mathrm{H}^k_{\mathrm{dR}}(X;\mathbf{C}) \cong \bigoplus_{p+q=k} \mathrm{H}^{p,q}_\partial(X).$$

But we're not done yet: the final ingredient is Dolbeault's theorem, which states that

$$\mathrm{H}^{p,q}_\partial(X) \cong \mathrm{H}^{p,q}_{\mathrm{Dol}}(X).$$

2. Moving to algebraic geometry

Taking a step back, we see that we've proven a relationship between invariants derived from the smooth and holomorphic structures on our compact complex manifold. Whenever you prove something for compact complex things, it is natural to wonder whether the result can be formulated in the algebraic context. We can at least make sense of the words that go into the statement of the Hodge theorem.

Definition 2.1. Let k be a ring, and let A be a finitely presented k-algebra, so $A = k[t_1,\cdots,t_n]/(f_1,\cdots,f_m)$. Define the algebraic Kähler differentials $\Omega^1_{A/k}$ to be the quotient

$$\Omega^1_{A/k} = (Adt_1 \oplus \cdots \oplus Adt_n)/(df_1,\cdots,df_n).$$

One can show that this is independent of the presentation of A.

For example, if X is the affine curve cut out in $\mathbf{A}^2$ by $y^2 = x^3 + ax + b$, then $\Omega^1_{X/k}$ is the quotient of $\mathcal{O}_X dx \oplus \mathcal{O}_X dy$ by the relation

$$2ydy = d(y^2) = d(x^3 + ax + b) = (3x^2 + a)dx.$$

One can extend the definition of $\Omega^1_{A/k}$ to schemes of finite type over k; if X is such a scheme, then $\Omega^1_{X/k}$ is a sheaf of abelian groups on X. Importantly, we make no assumptions on k.

Just as in the smooth setting, we can define $\Omega^i_{X/k} = \wedge^i \Omega^1_{X/k}$, and one can define differentials

$$d : \Omega^i_{X/k} \to \Omega^{i+1}_{X/k},$$

which satisfy the usual relation $d^2 = 0$. This defines a chain complex

$$\Omega^0_{X/k} = \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/k} \xrightarrow{d} \Omega^2_{X/k} \to \cdots$$

of sheaves of abelian groups on X.

Definition 2.2. The complex above is denoted $\Omega^\bullet_X$, and is called the *algebraic de Rham complex* of X.

There is a way to take the cohomology of $\Omega^\bullet_X$ (called "hypercohomology"), and therefore to make the following definition:

Definition 2.3. Let X be a scheme of finite type over k. Define $\mathrm{H}^i_{\mathrm{dR}}(X)$ to be the ith cohomology of $\Omega^\bullet_X$; this is the "algebraic" de Rham cohomology of X.

Let's now assume that $k = \mathbf{C}$ is the field of complex numbers. Then, the complex points $X(\mathbf{C})$ of X is a topological space which can be equipped with the *analytic topology*. For instance, if X is the scheme cut out by $y^2 = x^3 + ax + b$, as above, then $X(\mathbf{C})$ is the subspace of $\mathbf{C}^2$ cut out by this elliptic curve, and the analytic topology allows us to view it as a compact Riemann surface.

We can then ask:

Question 2.4. Suppose X is defined over $\mathbf{C}$. How does $\mathrm{H}^i_{\mathrm{dR}}(X)$ compare with $\mathrm{H}^i_{\mathrm{dR}}(X(\mathbf{C}))$?

Before answering this, let us try to work out an example. Consider $X = \mathbf{A}^1$, so $X(\mathbf{C}) = \mathbf{C}$. Then $\mathrm{H}^i_{\mathrm{dR}}(X(\mathbf{C})) = 0$ for $i > 0$, and is $\mathbf{C}$ for $i = 0$. What is $\mathrm{H}^i_{\mathrm{dR}}(\mathbf{A}^1)$? If t is a coordinate on $\mathbf{A}^1$, then the de Rham complex goes

$$\mathbf{C}[t] = \Omega^0_{\mathbf{A}^1} \xrightarrow{d} \Omega^1_{\mathbf{A}^1} = \mathbf{C}[t]dt.$$

Of course, everything is in the kernel of d on $\Omega^1_{\mathbf{A}^1}$. Now suppose $f(t)dt \in \mathbf{C}[t]dt$, and write $f(t) = \sum a_n t^n$. Define

$$g(t) = \sum \frac{a_n}{n+1} t^{n+1} \in \mathbf{C}[t];$$

then, it is easy to see that $dg(t) = f(t)dt$. This implies that $\mathrm{H}^i_{\mathrm{dR}}(\mathbf{A}^1) = 0$ for $i > 0$. What about $i = 0$? Well, if $f(t) \in \mathbf{C}[t]$ is such that $df = 0$, then f must be a constant, so $\mathrm{H}^0_{\mathrm{dR}}(\mathbf{A}^1) \cong \mathbf{C}$. In other words, the algebraic de Rham cohomology of $\mathbf{A}^1$ and the topological de Rham cohomology of $\mathbf{A}^1(\mathbf{C}) = \mathbf{C}$ are the same. This is true in general:

Theorem 2.5 (Grothendieck)**.** *If X is a smooth variety over $\mathbf{C}$, then $\mathrm{H}^i_{\mathrm{dR}}(X) \cong \mathrm{H}^i_{\mathrm{dR}}(X(\mathbf{C}))$.*

Grothendieck's theorem suggests a natural question:

Question 2.6. Suppose X is a smooth variety over $\mathbf{C}$. Is there an analogue of the Hodge decomposition for $\mathrm{H}^i_{\mathrm{dR}}(X)$?

In the analytic situation, the Hodge decomposition related $\mathrm{H}^i_{\mathrm{dR}}(X(\mathbf{C}))$ and $\mathrm{H}^q(X(\mathbf{C}); \Omega^{p,\bullet}_{X(\mathbf{C})})$. While it's not possible to define the "$\overline{\partial}$" operator in the algebro-geometric context, we can nonetheless just declare:

Definition 2.7. Suppose X is a smooth variety over a field k. Define[2] $\mathrm{H}^{p,q}_{\mathrm{Dol}}(X)$ to be $\mathrm{H}^q(X; \Omega^p_{X/k})$. Note that this does not need $k = \mathbf{C}$.

To phrase the Hodge decomposition in the algebraic context, it is useful to think about a big grid of abelian groups which is $\mathrm{H}^{p,q}_{\mathrm{Dol}}(X)$ in the (p,q)th slot. These groups fit together into a structure known as a spectral sequence, which I will not attempt to describe here. It's symbolically denoted

$$E_1^{p,q} = \mathrm{H}^{p,q}_{\mathrm{Dol}}(X) \Rightarrow \mathrm{H}^{p+q}_{\mathrm{dR}}(X). \tag{2.1}$$

There's a similar spectral sequence in the analytic setting. In the analytic setting, the Hodge decomposition can be phrased as the statement that the ith de Rham cohomology of $X(\mathbf{C})$ is the direct sum of all the groups on the diagonal line $p + q = i$. In spectral sequence lingo, this can be rephrased as saying that the above spectral sequence "degenerates at the E_1-page". We can therefore ask whether the spectral sequence in the algebraic setting also "degenerates at the E_1-page"; the answer is yes, and is essentially a restatement of the Hodge theorem:

Theorem 2.8. *If X is a smooth and proper variety over $\mathbf{C}$, then the spectral sequence (2.1) degenerates at the E_1-page.*

[2]Some care needs to be taken with this definition, but we won't worry about that here.

3. New and fertile land

If you trust that I'm not misleading you with this spectral sequence stuff, then you can rightly wonder: all the objects in (2.1) seem to make sense over any field, not just $\mathbf{C}$. Does this spectral sequence still degenerate? This is *a priori* a rather outlandish question to ask, because when our base field was $\mathbf{C}$, this was related to a lot of über-analytic stuff like the Laplacian. But if our base field is something like $\mathbf{F}_p$, what can we even compare de Rham cohomology to?

Let's try to revisit our calculation of the de Rham cohomology of $\mathbf{A}^1$ over $\mathbf{C}$, and see what goes wrong if we work over $\mathbf{F}_p$ instead. In this case, the de Rham complex goes

$$\mathbf{F}_p[t] = \Omega^0_{\mathbf{A}^1} \xrightarrow{d} \Omega^1_{\mathbf{A}^1} = \mathbf{F}_p[t]dt.$$

Again, everything is in the kernel of d on $\Omega^1_{\mathbf{A}^1}$. But what's the image? This can be determined via the following fundamental computation:

$$\boxed{d(t^p) = pt^{p-1}dt = 0.}$$

The image of $d : \mathbf{F}_p[t] \to \mathbf{F}_p[t]dt$ is spanned by the images of $d(t^i)$ for $i \geq 1$; but these vanish whenever $p|i$. Therefore,

$$\mathrm{H}^1_{\mathrm{dR}}(\mathbf{A}^1_{\mathbf{F}_p}) \cong t^{p-1}\mathbf{F}_p[t^p]dt.$$

Similarly, one sees from the fundamental computation that

$$\mathrm{H}^0_{\mathrm{dR}}(\mathbf{A}^1_{\mathbf{F}_p}) \cong \mathbf{F}_p[t^p].$$

There's a suggestive way to rephrase this computation. Define $(\mathbf{A}^1)^{(p)}$ to be $\operatorname{Spec} \mathbf{F}_p[t^p]$. Then the de Rham complex of $(\mathbf{A}^1)^{(p)}$ looks like

$$\mathbf{F}_p[t^p] = \Omega^0_{(\mathbf{A}^1)^{(p)}} \xrightarrow{d} \Omega^1_{(\mathbf{A}^1)^{(p)}} = \mathbf{F}_p[t^p]dt^p,$$

where we regard dt^p as a formal symbol. Then:

Lemma 3.1. *There is an isomorphism*

$$\Omega^i_{(\mathbf{A}^1)^{(p)}} \xrightarrow{\cong} \mathrm{H}^i_{\mathrm{dR}}(\mathbf{A}^1_{\mathbf{F}_p});$$

when $i = 1$, this sends dt^p to $t^{p-1}dt^p$.

This can be generalized to arbitrary varieties: if X is a scheme over a field k of characteristic $p > 0$, define $X^{(p)}$ to be the fiber product of $X \to \operatorname{Spec}(k)$ along the Frobenius $F : \operatorname{Spec}(k) \to \operatorname{Spec}(k)$. This is known as the Frobenius twist of X. Then:

Theorem 3.2 (Cartier isomorphism)**.** *If X is a smooth scheme over $\mathbf{F}_p$, then there is an isomorphism*

$$\Omega^i_{X^{(p)}} \xrightarrow{\cong} \mathrm{H}^i_{\mathrm{dR}}(X).$$

This is really wacky behavior: the affine line is as far from being contractible as it can be — its de Rham cohomology looks like the de Rham complex of its Frobenius twist! Deligne and Illusie had the fantastic idea of using this wackiness to their advantage: they (roughly) showed that under some conditions on X, the Cartier isomorphism can actually be lifted to an equivalence of *complexes*. More precisely, the de Rham cohomology $\mathrm{H}^i_{\mathrm{dR}}(X)$ is the ith cohomology of the de Rham complex $\Omega^\bullet_X$, and we can regard $\Omega^i_{X^{(p)}}$ as the cohomology of the silly-looking complex

$$\Omega^0_{X^{(p)}} \xrightarrow{0} \Omega^1_{X^{(p)}} \xrightarrow{0} \Omega^2_{X^{(p)}} \to \cdots,$$

where all the differentials are zero. We can denote the former complex by $(\Omega^\bullet_X, d)$, and the latter complex by $(\Omega^\bullet_{X^{(p)}}, 0)$. The Cartier isomorphism tells us that the cohomologies of this complex are the same, but this does not mean that the complexes themselves are equivalent. However:

Theorem 3.3 (Deligne-Illusie). *Suppose that X is a smooth and proper variety over $\mathbf{F}_p$ such that:*

- *the defining equations of X can be lifted to $\mathbf{Z}/p^2$; and*
- $\dim(X) < p$.

Then there is an equivalence[3] *between $(\Omega^\bullet_X, d)$ and $(\Omega^\bullet_{X^{(p)}}, 0)$.*

Moreover, this implies that the spectral sequence (2.1) for X degenerates at the E_1-page.

They *also* showed that once you have this result, you can deduce that the spectral sequence (2.1) degenerates for any smooth and proper variety defined over $\mathbf{C}$. In other words, their result implies the Hodge decomposition for the de Rham cohomology of a smooth and proper variety defined over $\mathbf{C}$! This should be shocking: the Hodge decomposition over $\mathbf{C}$ relied on a lot of heavy tools from analysis — but *none* of that appears in characteristic p.

4. Vista: categorifying the Hodge decomposition

We can try to categorify the Hodge decomposition. What does this mean? Suppose X is a complex manifold or a scheme. Then $\mathrm{H}^i_{\mathrm{dR}}(X; \mathbf{C})$ is the ith cohomology of the complex $(\Omega^\bullet_X, d)$, while $\bigoplus_{p+q=i} \mathrm{H}^q(X; \Omega^p_X)$ can be understood (in a precise sense) as the cohomology of the complex $(\Omega^\bullet_X, 0)$. We may therefore try to construct some equivalence of categories between "modules" over $(\Omega^\bullet_X, d)$ and $(\Omega^\bullet_X, 0)$.

It's convenient to fix a parameter $\lambda \in \mathbf{C}$, and consider the complex $(\Omega^\bullet_X, \lambda d)$, which interpolates between $(\Omega^\bullet_X, d)$ and $(\Omega^\bullet_X, 0)$ (by setting $\lambda = 1, 0$, respectively). What could we mean by a module over $(\Omega^\bullet_X, \lambda d)$? The map λd is entirely determined by the map $\lambda d : \mathcal{O}_X \to \Omega^1_X$, so we might try to understand "modules over $\lambda d : \mathcal{O}_X \to \Omega^1_X$". This can be done via the following definition:

Definition 4.1. A λ-connection on a $\mathcal{O}_X$-module $\mathcal{E}$ is a map $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$ such that if $f \in \mathcal{O}_X$ and $s \in \mathcal{E}$, then

$$\nabla(fs) = f\nabla(s) + \lambda df \otimes s.$$

It's said to be *flat* if $\nabla^2 = 0$, just as with flat connections.

If $\lambda = 1$, then this is just a flat connection on $\mathcal{E}$, while if $\lambda = 0$, then this is a $\mathcal{O}_X$-linear map $\mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$ which squares to zero. Such a map is known as a "Higgs field" on $\mathcal{E}$.

One may then say that a "categorification" of the Hodge theorem would be an equivalence of categories between certain modules with 1-connections on X (i.e., modules with flat connection) and certain modules with flat 0-connections on X. Such a result has been proven; it is known as the *nonabelian Hodge correspondence*, and combines deep work of many people (Narasimhan-Seshadri, Donaldson, Uhlenbeck-Yau, Beilinson-Deligne, Hitchin, Simpson, Siu, Sampson, Corlette, and Deligne). Roughly, the idea is to interpolate between 1-connections and 0-connections by defining a notion of "harmonic bundle", much like (in fact, generalizing) the use of harmonic forms in proof of the Hodge decomposition. If X is a Riemann surface, then these are essentially triples $(\mathcal{E}, \nabla, \theta)$ where ∇ is a flat connection on $\mathcal{E}$, and $\theta : \mathcal{E} \to \mathcal{E} \otimes \Omega^{1,0}_X$ is a "holomorphic" Higgs field, which solve the *Hitchin equations*: if $\widetilde{\nabla} = \nabla + \phi + \phi^*$, then

$$F(\widetilde{\nabla}) = 0, \ \nabla^{(0,1)}(\phi) = 0.$$

In other words, $\widetilde{\nabla}$ is flat, and ϕ is horizontal with respect to the $(0,1)$-component of ∇. These equations were obtained by Hitchin by considering the "self-dual Yang-Mills" equations on the 4-manifold $X \times \mathbf{R}^2$. The reason these equations are relevant to Hodge theory stems from the fact that the Yang-Mills equations are nonabelian generalizations of Laplace's equation.

[3] I'm glossing over a few things here. First, the word "equivalence" really means "quasi-isomorphism". Second, the result compares the pushforward of $(\Omega^\bullet_X, d)$ along the relative Frobenius $X \to X^{(p)}$ and $(\Omega^\bullet_{X^{(p)}}, 0)$.

Interestingly, there is *also* a comparison between certain modules with 1-connections on X (i.e., modules with flat connection) and certain modules with flat 0-connections on X when X is a smooth scheme over $\mathbf{F}_p$, due to Ogus and Vologodsky. This categorifies the Deligne-Illusie theorem. In this setting, one also has an analogue of "harmonic bundles". These are certain triples $(\mathcal{E}, \nabla, \phi)$ defined on X (which we will assume is a curve), where $\mathcal{E}$ is a bundle on X with flat connection ∇, and $\phi : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$ is a Higgs field. These triples solve the following analogue of Hitchin's equations: if $\widetilde{\nabla} = \nabla - \phi$, then

$$\psi(\widetilde{\nabla}) = 0, \; (\widetilde{\nabla} \otimes \nabla^{\mathrm{can}})(\psi(\nabla)) = 0.$$

Here, ψ is the "p-curvature" of a connection, and ∇^{can} denotes the "canonical connection", neither of which we will attempt to define. The similarity between these equations and Hitchin's equations is tantalizing: although these two areas seem to have gone their own ways, there are still some questions which remain that I don't know how to answer. Here's one:

Question 4.2. Is there a way to deduce (special cases of) the nonabelian Hodge correspondence in characteristic zero from the nonabelian Hodge correspondence in characteristic p, just as how Deligne and Illusie deduce the Hodge theorem over $\mathbf{C}$ from their results over field of positive characteristic?

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